Topological K-theory for discrete groups and index formula

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Reference:

- Paulo Carrillo-Rouse, Bai-Ling Wang, Hang Wang
  Topological K-theory for discrete groups and index theory
  ArXiv 2020.

Outline:

- Topological K-theory for discrete groups and assembly map;
- K-homology and assembly map;
- Comparing topological K-theory and K-homology;
- What is good about topological K-theory for discrete groups.

I. Topological K-theory for discrete groups and assembly map

Motivation

Let Γ be a discrete group.

Question

$$K_*(C_r^*(\Gamma)) =?$$

In 1982, Baum-Connes proposed that a "topological K-theory for Γ" can be mapped to K-theory

$$\mu : K_{top}^*(\Gamma) \to K_*(C_r^*(\Gamma))$$

and conjectured that the "assembly map" μ is an isomorphism.

Topological K-theory for Γ

For a discrete group Γ, an element of $K^*_{top}(\Gamma)$ is represented by cycles (M, x) where

- M is a Γ-proper cocompact spin^c manifold;
- $x \in K^*_\Gamma(M) := K_*(C_0(M) \rtimes \Gamma)$

subject to relations

$$(M, x) \sim (N, f_! x).$$

where $f_! : K^*_\Gamma(M) \to K^*_\Gamma(N)$ is the pushforward map induced from a Γ-equivariant and K-oriented smooth map $f : M \to N$.

The topological K-theory of Γ is defined by

$$K^*_{top}(\Gamma) := \varinjlim_{f_!} K^*_\Gamma(M).$$

Theorem (Baum-Connes, 'Carrillo-Rouse'-Wang)

There is a well-defined assembly map $\mu : K^*_{top}(\Gamma) \to K_*(C^*_r(\Gamma))$

$$\mu([M, x]) := \pi_!(x).$$

Here, $\pi : M \to pt$ *and* $\pi_! : K^*_\Gamma(M) \to K^*_\Gamma(pt) \simeq K_*(C^*_r(\Gamma))$.

Pushforward map on K-theory

- Let $f : M \to N$ be a Γ-equivariant K-oriented map.
- Let $T_f : TM \oplus f^*TN \to M$ and then T_f^* has a spin^c-structure.

The pushforward map $f_! : K_\Gamma^{*-r_f}(M) \to K_\Gamma^*(N)$ is defined by the compositions:

- $Th : K_\Gamma^{*-r_f}(M) \to K_\Gamma^*(T_f^*)$
- $F : K_\Gamma^*(T_f^*) \to K_\Gamma^*(T_f)$
- $(e_{0,*})^{-1} : K_\Gamma^*(T_f) \to K_\Gamma^*(D_f)$
- $e_{1,*} : K_\Gamma^*(D_f) \to K_\Gamma^*(M \times M \times N)$
- $M : K_\Gamma^*(M \times M \times N) \to K_\Gamma^*(N)$

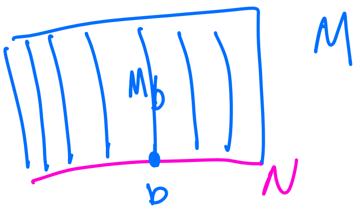
The deformation groupoid

$$D_f : TM \oplus f^*TN \bigsqcup (M \times M \times N) \times (0,1] \rightrightarrows f^*TN \bigsqcup (M \times N) \times (0,1]$$

is obtained from the normal groupoid associated to the Γ-map

$$M \xrightarrow{\Delta \times f} M \times M \times N \qquad x \mapsto (x, x, f(x)).$$

If Γ trivial $\quad M \to N$ is a submersion



K-theory pushforward is a family index map.

$$K^0(M) \longrightarrow K^0(N)$$
$$[E] \longmapsto \{\mathrm{ind}\, D_{b,E}\}_{b\in N}$$

If Γ trivial, $N = pt$,

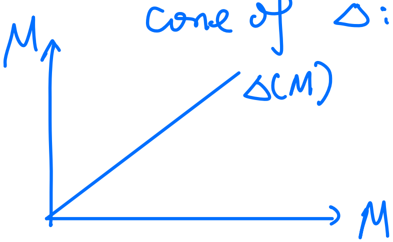
$$K^0(M) \longrightarrow K^0(pt) \cong \mathbb{Z} \qquad [E] \longmapsto \mathrm{ind}\, D_E$$

normal bundle of $\Delta(M)$ in $M\times M$

$$M \rightsquigarrow T^*M \rightsquigarrow TM \xleftarrow{ev_0} \mathcal{T} \xrightarrow{ev_1} M\times M \rightsquigarrow pt$$

$$C_0(T^*M) \cong C^*(TM) \qquad\qquad C^*(M\times M) \cong K(L^2(M))$$

$\mathcal{T}$: Connes' tangent groupoid, deformation to the normal cone of $\Delta: M \to M\times M$



$$\mathcal{T} = TM\times\{0\} \sqcup M\times M\times(0,1]$$
$$(x,\xi) \qquad (x, y, t)$$

$$(x_n, y_n, t_n) \to (x,\xi) \iff x_n - y_n, t_n \to 0 \ \& \ \lim_{n\to\infty} \frac{x_n - y_n}{t_n} = \xi$$

Example: pushforward as an index

When $N = pt$ and Γ is trivial, $f_! : K^{*-r}(M) \to K^*(pt)$ is given by

$$K^{*-r}(M) \to K^*(T^*M) \to K_*(C_0(T^*M)) \to$$
$$K_*(C_r^*(TM)) \leftarrow K_*(C_r^*(\mathcal{T})) \to K_*(C_r^*(M \times M)) \to K^*(pt) \simeq \mathbb{Z}.$$

Here $\mathcal{T} = TM \times \{0\} \bigsqcup M \times M \times (0,1]$ is Connes' tangent groupoid fitting in

$$0 \to C_0((0,1], C_r^*(M \times M)) \to C_r^*(\mathcal{T}) \stackrel{e_0}{\to} C_r^*(TM) \to 0.$$

and gives rise to

$$e_{1*} \circ e_{0*}^{-1} : K_*(C_r^*(TM)) \simeq K_*(C_r^*(\mathcal{T})) \to K_*(C_r^*(M \times M)).$$

Connes' tangent groupoid leads to the analytic index of Atiyah-Singer:

$$K^{*-r}(M) \to K^*(T^*M) \to K_*(C_r^*(M \times M)) \simeq \mathbb{Z} :$$
$$[E] \mapsto [\sigma_{D_E}] \mapsto \text{ind } D_E.$$

II. K-homology and assembly map

Alternative description of assembly map

In 1994, Baum-Connes-Higson reformulated the Baum-Connes assembly map using "analytic K-homology":

$$\mu : K_*^{\Gamma}(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma)).$$

where $\underline{E}\Gamma$ is universal space of proper actions by Γ in the sense that for any Γ-proper cocompact manifold X, there exists a Γ-equivariant continuous map

$$f : X \to \underline{E}\Gamma.$$

Analytic K-homology

- Atiyah formulated analytic K-homology for closed manifolds M as dual theory of K-theory using elliptic operators D

$$[D] \in K_0(M).$$

- Brown-Douglas-Fillmore used extension to formulate K-homology

$$\partial : K_1(\overset{f}{C(S^1)}) \to K_0(\overset{\text{ind } T_f}{\mathcal{K}(H)}) \cong \mathbb{Z}$$

$$[0 \to \mathcal{K}(H) \to \mathcal{T} \to C(S^1) \to 0] \in K_1(S^1)$$

- K-homology is generalized to Kasparov's KK-theory $KK(A, B)$ using extensions

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$

or abstract elliptic operators

$$(\mathcal{E}_B, \phi : A \to \mathcal{L}(\mathcal{E}_B), F \in \mathcal{L}(\mathcal{E}_B)).$$

Assembly map

The equivariant K-homology of the universal proper space is defined by

$$K_*^\Gamma(\underline{E}\Gamma) := \lim_{X \subset \underline{E}\Gamma} K_*^\Gamma(X)$$

where the inductive limit is with respect to

$$f_* : K_*^\Gamma(X) \to K_*^\Gamma(Y)$$

$C_0(Y) \xrightarrow{f^\#} C_0(X) \longrightarrow B(H)$

induced by a smooth equivariant map between Γ proper cocompact spaces

$$f : X \to Y.$$

The assembly map

$$\mu : K_*^\Gamma(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma)) \qquad [D_X] \to \mathrm{ind}_\Gamma D_X$$

is given by the generalized Fredholm index of an abstract elliptic operator D_X associated to a proper cocompact space X.

III. Comparing topological K-theory with K-homology

Geometric K-homology

Baum-Douglas developed the geometric K-homology for a closed manifold or a finite CW complex X. It is represented by geometric cycles (M, f, x) where

- M is a closed spin^c manifold;
- $f : M \to X$ is a continuous map;
- $x \in K^0(X)$, represented by a finite dimensional complex bundle over X

subject to equivalence by bordisms and vector bundle modifications.
A geometric cycle maps to analytic K-homology

$$K_*^{top}(X) \to K_*(X) \qquad [M, f, x] \mapsto [D_{M, f^*x}]$$

where D_M is the spin^c Dirac operator on M.

Geometric and analytic K-homology

Baum-Higson-Schick showed that the geometric and analytic K-homology are isomorphic:

$$K_*^{top}(X) \cong K_*(X),$$

and they also generalized to the case of X carrying a proper cocompact discrete group action.

Theorem (Baum-Higson-Schick)

There is an isomorphism

$$\lim_{f_*} K_*^{\Gamma}(M) \to \lim_{X \subset \underline{E}\Gamma} K_*^{\Gamma}(X) =: K_*^{\Gamma}(\underline{E}\Gamma).$$

where M is Γ-spin^c, proper, cocompact and $f : M \to N$ is a Γ-equivariant smooth map.

Functoriality and Poincaré duality

Proposition (CR-W-W)

Let M, M' be proper cocompact spin^c manifolds, $g : M \to M'$ is a Γ-equivariant smooth map. The following diagram commutes:

$$\begin{array}{ccc} K_*^{\Gamma}(M) & \xrightarrow{g_*} & K_*^{\Gamma}(M') \\ \downarrow PD_M & & \downarrow PD_{M'} \\ K^*_{\Gamma}(M) & \xrightarrow[g_!]{} & K^*_{\Gamma}(M') \end{array}$$

where

$$PD_M : K_*^{\Gamma}(M) \to K^*_{\Gamma}(M) \qquad [N, f, x] \mapsto f_!(x)$$

is the Poincaré duality (following Baum-Higson-Schick).

The commutative diagram lead to Poincaré duality

$$\lim_{f_*} K_*^{\Gamma}(M) \to \lim_{f_!} K^*_{\Gamma}(M) =: K^*_{top}(\Gamma).$$

Comparing LHS of assembly map

Corollary (CR-W-W)

Let Γ *be a countable discrete group. There is a unique isomorphism*

$$K^*_{top}(\Gamma) \to K^\Gamma(\underline{E}\Gamma)$$

fitting in the commutative diagram:

$$\begin{array}{ccc} \lim_{f_*} K^\Gamma_*(M) & \xrightarrow[\cong]{BHS} & \lim_{X \subset \underline{E}\Gamma} K^\Gamma_*(X) \\ \cong \downarrow PD & & \downarrow \cong \\ K^*_{top}(\Gamma) & \longrightarrow & K^\Gamma_*(\underline{E}\Gamma). \end{array}$$

Other models of K-homology and assembly map

Let X be a Γ-proper cocompact space.

- (Higson-Roe "Analytic K-theory")

$$K_*^\Gamma(X) \cong K_{*+1}(D^*(X)^\Gamma / C^*(X)^\Gamma)$$

- (Willett-Yu "Higher index theory")

$$K_*^\Gamma(X) \cong K_*(C_L^*(X)^\Gamma)$$

Two pictures of assembly map:

- Generalized Fredholm operator picture

$$K_*^\Gamma(X) \cong K_{*+1}(D^*(X)^\Gamma / C^*(X)^\Gamma) \to K_*(C^*(X)^\Gamma)$$

- Deformation groupoid, local-global picture

$$K_*^\Gamma(X) \cong K_*(C_L^*(X)^\Gamma) \to K_*(C^*(X)^\Gamma)$$

IV. What is good about topological K-theory for discrete groups?

Index formula is naturally encoded in topological K-theory for discrete groups (K-theory pushforward map) by taking Chern character.

An example of index formula

Let M be a closed complex manifold, $E \to M$ be a holomorphic vector bundle, and $\mathcal{O}(E)$ be holomorphic sections.
Consider the Dolbeault complex of E:

$$\overline{\partial} : 0 \to \Omega^{0,0}(E) \to \Omega^{0,1}(E) \to \cdots \to \Omega^{0,n}(E) \to 0.$$

Let

$$\chi(M, E) := \sum_q (-1)^q \dim H^q(M, \mathcal{O}(E)).$$

Theorem (Riemann-Roch, Atiyah-Singer)
$D := \overline{\partial} + \overline{\partial}^* : \Omega^{0,even}(E) \to \Omega^{0,odd}(E)$ *is Fredholm and*

$$\operatorname{ind} D = \chi(M, E) = \int_M \mathrm{ch}(E)\mathrm{Td}(M).$$

Remark
Formula remains if “complex” is replaced by “spin$^{(c)}$.”

Chern character defect

The proof of the Atiyah-Singer index formula is essentially an observation of Chern character defect relative to the Thom isomorphisms as a generalization of Riemann-Roch formula.

Let

- N be the tubular neighborhood of M in $\mathbb{R}^N$ identified as the normal bundle;
- $f : M \to E := N \otimes \mathbb{C}$ the zero section;
- $f_!$ be Thom isomorphisms.

The diagram

$$\begin{array}{ccc} K^0(M) & \xrightarrow{f_!} & K^0(E) \\ \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch} \\ H^{even}(M) & \xrightarrow[f_!]{} & H_c^{even}(E) \end{array}$$

commutes up to the Todd class of M.

Delocalized Riemann-Roch theorem

Let Γ be a discrete group.

- Let M, N be proper cocompact Γ-manifolds;
- Let $f : M \to N$ be a Γ-equivariant K-oriented map, i.e., $TM \oplus f^*TN \to M$ is spin^c. Let $\dim TM \oplus f^*TN = r$.

Theorem (CR-W-W)

The diagram commutes:

$$\begin{array}{ccc} K_\Gamma^{*-r}(M) & \xrightarrow{f_!} & K_\Gamma^*(N) \\ \downarrow {\scriptstyle ch_{Td_M^\Gamma}} & & \downarrow {\scriptstyle ch_{Td_N^\Gamma}} \\ H_{\Gamma,deloc}^{*-r}(M) & \xrightarrow[f_!]{} & H_{\Gamma,deloc}^*(N). \end{array}$$

where for $E \to M$ a Γ-proper spin^c bundle,

$$\mathrm{ch}_{Td_E^\Gamma} : K_\Gamma^*(M) \to H_{\Gamma,deloc}^*(M) \qquad x \mapsto \mathrm{ch}_M^\Gamma(x) \wedge \mathrm{Td}^\Gamma(E)$$

is the "twisted delocalized Chern character".

Delicalized cohomology (Tu-Xu)

Let M be a proper Γ-manifold. Consider the periodic delocalized cohomology groups for $* = 0, 1 \; mod \, 2$:

$$H^*_{\Gamma, deloc}(M) := \bigoplus_{g \in \langle \Gamma \rangle^{fin}} \prod_{k = *, \, mod \, 2} H^k_c(M_g \rtimes \Gamma_g),$$

where

- $M_g = \{x \in M : x \cdot g = x\}$
- $\Gamma_g = \{h \in \Gamma : hg = gh\}$
- $M_g \rtimes \Gamma_g$ is the action groupoid
- $g \in \Gamma$ is a fixed, finite order element
- $\langle \Gamma \rangle^{fin}$ stands for the set of conjugacy classes of finite order elements of Γ.

Chern character (Tu-Xu)

Let M be a Γ-proper manifold. The Connes-Chern character

$$Ch : K_\Gamma^*(M) \longrightarrow HP_*(C_c^\infty(M \rtimes \Gamma)).$$

followed by the Tu-Xu isomorphism

$$HP_*(C_c^\infty(M \rtimes \Gamma)) \xrightarrow[\cong]{TX} H^*_{\Gamma,deloc}(M).$$

defines the Γ-Chern character:

$$ch_M^\Gamma : K_\Gamma^*(M) \longrightarrow H^*_{\Gamma,deloc}(M).$$

- Tu and Xu proved that ch_M^Γ is an isomorphism up to $\otimes \mathbb{C}$.
- For every $x \in K_\Gamma^*(M)$, we have a component decomposition

$$ch_M^\Gamma(x) = (ch_M^g(x))_{g \in \langle \Gamma \rangle^{fin}} \in \bigoplus_{g \in \langle \Gamma \rangle^{fin}} \left(\bigoplus_{k=*, \, mod\, 2} H_c^k(M_g \rtimes \Gamma_g) \right).$$

Delocalized *Todd*-class

Let $E \longrightarrow M$ be a Γ-proper spin^c vector bundle over M. One has the Thom isomorphism in equivariant K-theory (Kasparov)

$$Th : K^*_\Gamma(M) \longrightarrow K^{*+r_E}_\Gamma(E),$$

and for $g \in \Gamma$ of finite order the delocalised Thom isomorphism (Behrend-Ginot-Noohi-Xu)

$$Th_g : \bigoplus_{k=*,\, mod\, 2} H^k_c(M_g \rtimes \Gamma_g) \longrightarrow \bigoplus_{k=*,\, mod\, 2} H^{k+r_E}_{sch}(E_g \rtimes \Gamma_g),$$

The delocalized *Todd*-class

$$Td^\Gamma_E := (Td^E_g)_{g\in\langle\Gamma\rangle^{fin}} \in \bigoplus_{g\in\langle G\rangle^{fin}} \left(\bigoplus_{k=*,\, mod\, 2} H^k(E_g \rtimes \Gamma_g) \right)$$

where each class Td^E_g is the unique class satisfying

$$Th_g(ch^g_M(x) \wedge Td^E_g) = ch^g_E(Th(x)) \qquad \forall x \in K^*_\Gamma(M).$$

Twisted Γ-Chern character & cohomological pushforward

Consider the twisted Γ-Chern character morphism associated to E

$$ch_{Td_E^\Gamma} : K_\Gamma^*(M) \longrightarrow H^*_{\Gamma,deloc}(M),$$

given by

$$ch_{Td_E^\Gamma}(x) := ch_M^\Gamma(x) \wedge Td_E^\Gamma.$$

Analogously, for $f : M \to N$ we have cohomological pushforward

$$f_! : H^{*-r_f}_{\Gamma,deloc}(M) \to H^*_{\Gamma,deloc}(N)$$

by composing

- $Th : H^{*-r_f}_{\Gamma,deloc}(M) \to H^*_{\Gamma,deloc}(T_f^*)$
- $(TX)^{-1} : H^*_{\Gamma,deloc}(T_f^*) \to HP_*(\mathcal{S}(T_f^* \rtimes \Gamma))$
- $F : HP_*(\mathcal{S}(T_f^* \rtimes \Gamma)) \to HP_*(\mathcal{S}(T_f \rtimes \Gamma))$
- $(e_{0,*})^{-1} : HP_*(\mathcal{S}(T_f \rtimes \Gamma)) \to HP_*(\mathcal{S}(D_f \rtimes \Gamma))$
- $e_{1,*} : HP_*(\mathcal{S}(D_f \rtimes \Gamma)) \to HP_*(C_c^\infty((M \times M \times N) \rtimes \Gamma))$
- $M : HP_*(C_c^\infty((M \times M \times N) \rtimes \Gamma)) \to HP_*(C_c^\infty(N \rtimes \Gamma))$
- $TX : HP_*(C_c^\infty(N \rtimes \Gamma)) \to H^*_{\Gamma,deloc}(N).$

Main result

Theorem (CR-W-W)

Let $f : M \longrightarrow N$ be a Γ-equivariant, K-oriented, smooth map between two Γ-proper cocompact manifolds M and N. The following diagram commutes:

$$\begin{array}{ccc} K_\Gamma^{*-r}(M) & \xrightarrow{f_!} & K_\Gamma^*(N) \\ \downarrow{\scriptstyle ch_{Td_M^\Gamma}} & & \downarrow{\scriptstyle ch_{Td_N^\Gamma}} \\ H_{\Gamma,deloc}^{*-r}(M) & \xrightarrow[f_!]{} & H_{\Gamma,deloc}^*(N). \end{array}$$

When Γ is trivial, this reduces to the Riemann-Roch theorem

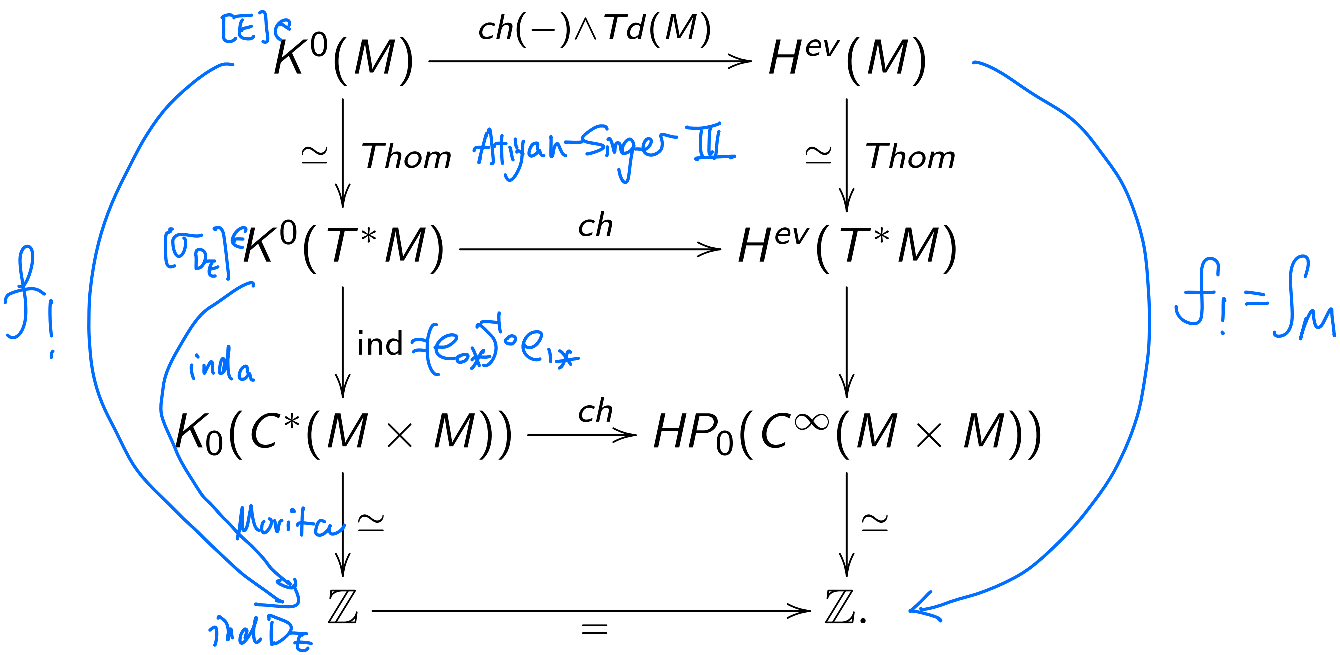
$$ch(f_!(E)) \wedge \mathrm{Td}(N) = f_!(\mathrm{ch}(E) \wedge \mathrm{Td}(M)).$$

If N is in addition a point, this is

$$f_!(E) = f_!(\mathrm{ch}(E) \wedge \mathrm{Td}(M)) = \int_M \mathrm{ch}(E) \wedge \mathrm{Td}(M) = \mathrm{ind}\, D_E.$$

Example: N is a point and Γ is trivial

For $f : M \to pt$ K-oriented, $f_! : K^*(M) \to K^*(pt)$ is given by the analytic index of D_M.



When M is closed spin^c and D is Dirac, commutativity of the diagram implies the Riemann-Roch theorem

$$\text{ind } D_E = \int_M \mathrm{ch}(E) \wedge \mathrm{Td}(M).$$

Assembled Chern character

For a discrete group Γ and for $* = 0, 1 \; mod\, 2$ we define the delocalized cohomology for discrete groups

$$H^*_{top}(\Gamma) = \varinjlim_{f_!} H^*_{\Gamma, deloc}(M).$$

Theorem (CR-W-W)

For a discrete group Γ, there is a well-defined Chern character morphism

$$ch^{top} : K^*_{top}(\Gamma) \longrightarrow H^*_{top}(\Gamma)$$
$$ch^{top}([M, x]) = [M, ch^{\Gamma}_M(x) \wedge Td^{\Gamma}_M].$$

Furthermore, it is an isomorphism once tensoring with $\mathbb{C}$.

Pairing on the LHS of the assembly map

Theorem (CR-W-W)

One has a cohomological assembly map

$$H^*_{top}(\Gamma) \to HP_*(\mathbb{C}\Gamma)$$

where a cohomological formula is obtained from pairing with $\tau \in HP^*(\mathbb{C}\Gamma)$.

Remark

A commutative diagram of index formulas

$$\begin{array}{ccccc} K^*_{top}(\Gamma) & \xrightarrow{\mu} & K_*(C^*_r(\Gamma)) & & \\ \downarrow{\scriptstyle ch^{top}} & & & \searrow & \\ H^*_{top}(\Gamma) & \xrightarrow[\mu]{} & HP_*(\mathbb{C}\Gamma) & \xrightarrow[\tau]{} & \mathbb{C}. \end{array}$$

is expected.

Thank you!