Exactness and geometric properties of inverse semigroups

Diego Martínez – WWU Münster
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Non-commutativity in the North – Göteborgs Universitet

Based on joint work with Chyuan and Szakács
Outline

(1) Inverse semigroups
(2) Proper and right sub-invariant metrics
(3) Exactness vs. Yu’s property A
(4) Asymptotic dimension 0 vs. local AF
1. Inverse semigroups
Inverse semigroups and Wagner-Preston

**Definition**

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Remarks:

- **Bicyclic monoid**: $B = \langle a, a^* \mid a^*a = 1 \rangle = \{a^i a^j \mid i, j \geq 0\}$
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- \( E = \{ e \in S \mid e^2 = e \} = \{ s^* s \mid s \in S \} \) is commutative
- \( D_{s^* s} = s^* s \cdot S \) is the *domain of s*
- \( s: D_{s^* s} \to D_{ss^*}, \) where \( x \mapsto sx \) is a bijection
**Inverse semigroups and Wagner-Preston**

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**Induces** the **Wagner-Preston** representation $v: S \to \mathcal{I}(S)$:
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**Induces the **Wagner-Preston** representation $v: S \rightarrow \mathcal{I}(S)$:**

$$D_{(st)(st)^*} = s (D_{tt^*} \cap D_{s^*s})$$
Open bisections and partial order

**Remark:** the Wagner-Preston representation encapsulates the idea behind how we see these inverse semigroups:

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**Partial order:** \( s \geq t \iff \) there is some \( e \in E \) with \( se = t \),

\[ \iff t \text{ is a} \text{ restriction} \text{ of } s \iff st^* t = t \]
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**Partial order:** $s \geq t \iff$ there is some $e \in E$ with $se = t$,  
$\iff t$ is a **restriction** of $s \iff st^* t = t$

**Left regular representation:** $\nu: S \rightarrow B(\ell^2(S))$, where  
$$
\nu_s \delta_x = \begin{cases} 
\delta_{sx} & \text{if } x \in s^* s \cdot S \\
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**Reduced C*-algebra:** $C^*_r(S) := C^*(\{\nu_s\}_{s \in S}) \subset \mathcal{B}(\ell^2(S))$
Recall: Cayley graph construction $\sim G = \langle g_1^{\pm 1}, \ldots, g_n^{\pm 1} \mid \text{relations} \rangle$:

- Graph $\sim \text{Cay}(G, \{g_1, \ldots, g_n\}) := (V, E)$,
- Vertices $\sim V := G$
- Edges $\sim E := \{(x, g_i^{\pm 1}x) \mid x \in G \text{ and } i = 1, \ldots, n\}$.
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**Proposition (classical)**

The large scale geometry of the Cayley graph of $G$ does **not** depend on the generators.

**Goal**: coarse geometry of inverse semigroups and its relation with $C^*$-properties of $C_r^* (S)$
2. Proper and right sub-invariant metrics
Remark: we need to consider extended metric spaces:

\[ S := G \sqcup \{0\} \sim s \cdot 0 = 0 \cdot s = 0 = s^* \cdot 0 = 0 \cdot s^* \]

and, hence, there is a directed edge \( s \rightarrow 0 \sim d(s, 0) = \infty \)
Infinite distances, and why they are necessary

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Let \( d: S \times S \to [0, \infty] \). We say \( d \) *respects the components of \( S \) if*

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- \( s^* s = 0^* 0 = 0 \Rightarrow s = ss^* s = s0 = 0 \),
  and therefore \( \{0\} \subset S \) forms a component!
- \( (S, d) = \sqcup_{e \in E} (L_e, d|_{L_e}) \) are the **connected** components
- This allows for uncountable \( S \)
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Standing assumption: \( d \) respects the components of \( S \)
Proper and right sub-invariant metrics

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Let \( d: S \times S \to [0, \infty] \) be a metric. We say \( d \) is:

- **right sub-invariant** if \( d(sr, tr) \leq d(s, t) \) for all \( s, t, r \in S \)
- **proper** if for all \( r \geq 0 \) there is a finite \( F \subseteq S \) such that \( t \in Fs \) for all \( s, t \in S \) such that \( d(s, t) \leq r \).
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  - For instance, if \( S = \bigsqcup_{e \in E} G_e \), then \( (S, d) = \bigsqcup_{e \in E} (G_e, d|_{G_e}) \)
- \( S = \langle s_1, \ldots, s_k \mid \text{relations} \rangle \nleftrightarrow d \) is the path metric in \( \{\Lambda_e\}_{e \in E} \)
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  For instance, if $S = \sqcup_{e \in E} G_e$, then $(S, d) = \sqcup_{e \in E}(G_e, d_{|G_e})$
- $S = \langle s_1, \ldots, s_k \mid \text{relations} \rangle \twoheadrightarrow d$ is the path metric in $\{\Lambda_e\}_{e \in E}$
- If $d$ is proper, then $(S, d)$ has bounded geometry
  
  However, the converse is false!
Existence and uniqueness of these metrics

**Theorem (Chung, M. and Szakács - 22)**

Every countable inverse semigroup has a *proper* and *right sub-invariant* metric. Moreover, such a metric is *unique* up to bijective coarse equivalence.
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**Remark:** works for some **non-countable** semigroups…
as long as \( S = \langle F \cup E \rangle \), where \( F \) is **countable**

**For instance:** an action \( G \rhd \text{Cantor} \), where \( G \) is a discrete group, induces \( S = \text{Bis}(G \rhd \text{Cantor}) \) as **above**
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**Question:** what sort of metric spaces $(S, d)$ can we get?
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**Theorem (Chung, M. and Szakács - 22)**

Any $(X, d)$ of **bounded geometry** is a component of some inverse semigroup (that depends on $X$).
3. Exactness vs. Yu’s property A
### Metric spaces and property A

**Definition (Yu - 1999)**

$(X, d)$ has **property A** if for every $r, \varepsilon > 0$ there is 
$\xi: X \to \ell^1 (X)_1^+$ and $c > 0$ such that $\text{supp} (\xi_x) \subset B_c (x)$ and 
$\|\xi_x - \xi_y\|_1 \leq \varepsilon$ for every $x, y \in X$ such that $d(x, y) \leq r$
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- Property A generalizes amenability for groups (not in general)
- Non-property A groups are hard to come by
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**Theorem (Ozawa - 2000)**

Let $G$ be a countable group. TFAE:

1. $(G, d)$ has property A, where $d$ is proper and r.inv.
2. $\ell^\infty (G) \rtimes_r G$ is nuclear.
3. $C^*_r (G)$ is exact.
Theorem (Lledó, M. - 2021, and Alcides, M. - 2022)

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Proof: (i) $\Rightarrow$ (ii) given $\xi: S \to \ell^1 (S)_1^+$ the diagram

$$\mathcal{R}_S \to \prod_{x \in S} M_{Bc(x)} \subset \ell^\infty (S) \otimes M_q \to \mathcal{R}_S$$

$$a \mapsto \left( p_{Bc(x)} \ a \ p_{Bc(x)} \right)_{x \in S} \leadsto (b_x)_{x \in S} \mapsto \sum_{x \in S} \xi_x^* b_x \xi_x$$

can be shown to be an approximation of $\text{id}: \mathcal{R}_S \to \mathcal{R}_S$

(ii) $\Rightarrow$ (iii) is clear, while

(iii) $\Rightarrow$ (i) is based on $\ell^\infty (S) \rtimes_r S \cong C^*_u (S, d)$
4. Asymptotic dimension 0 vs. local AF algebras
Semigroups of asymptotic dimension 0

Recall: \( \text{asdim}(X, d) = 0 \) is an analog for being a Cantor set

**Definition**

\[
\text{asdim}(X, d) = 0 \text{ if for every } r \geq 0, X \text{ has a partition } \mathcal{U} \text{ such that }
\inf_{U \neq V \in \mathcal{U}} d(U, V) \geq r \text{ and } \sup_{U \in \mathcal{U}} \text{diam}(U) < \infty
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Semigroups of asymptotic dimension 0

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**Question:** when does $S$ have asymptotic dimension 0?

**Answers:**

- If $S$ is finite then $\text{asdim}(S) = 0$
Semigroups of asymptotic dimension 0

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**Answers:**

- If \( S \) is finite then \( \mathrm{asdim}(S) = 0 \)
- If \( S \) is fin. gen., then \( S \) finite iff \( \mathrm{asdim}(S) = 0 \)
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- If \( S \) is finite then \( \text{asdim}(S) = 0 \)
- If \( S \) is fin. gen., then \( S \) finite iff \( \text{asdim}(S) = 0 \)
- If we add new generators \( S = \langle \{t_1, \ldots, t_n\} \cup \{s_1, \ldots, s_m\} \rangle \) then \[
\sup_{j=1, \ldots, n} d(t_j^* t_j, t_j) < \inf_{i=1, \ldots, m} d(s_i^* s_i, s_i),
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and that doesn’t increase the asymptotic dimension
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  \[ \sup_{j=1,\ldots,n} d(t_j^*t_j, tj) < \inf_{i=1,\ldots,m} d(s_i^*s_i, si), \]
  and that doesn’t increase the asymptotic dimension
- Hence, \( \text{asdim}(S) = 0 \) when \( S \) is **locally finite**
### Theorem (Chung, M. and Szakács - 22)

Let $S$ be an inverse semigroup. TFAE:

(i) $S$ is locally finite.

(ii) $\text{asdim}(S, d) = 0$, where $d$ is proper and $r$.inv.

(iii) $\ell^\infty(S) \rtimes_r S$ is local AF.

(iv) $\ell^\infty(S) \rtimes_r S$ is strongly quasidiagonal.

**Remark:** strongly quasidiagonal $\Rightarrow$ quasidiagonal
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Local AF algebras and quasidiagonality I

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Theorem (Chung, M. and Szakács - 22)

Let $S$ be an inverse semigroup. TFAE:

(i) $S$ locally has finite components.
(ii) $(S, d)$ is sparse, where $d$ is proper and r.inv.
(iii) $\ell^\infty(S) \rtimes_r S$ is quasidiagonal.
(iv) $\ell^\infty(S) \rtimes_r S$ is finite.
A bit about the proof:
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\( S \text{ locally finite} \Rightarrow \text{asdym} (S) = 0 \): sketched before
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- $S$ locally finite $\iff \text{asdym}(S) = 0$: $S$ is quasidiagonal
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\( S \text{ locally finite} \iff \text{asdym}(S) = 0 \):

\( S \text{ sparse} \Rightarrow \ell^\infty(S) \rtimes_r S \text{ is quasidiagonal} \):
Sometimes these classes coincide, i.e.,

take $G \curvearrowright X$, where $X$ is the Cantor set and $G$ discrete group,
then: $\text{Bis}(G \curvearrowright X)$ is \underline{locally finite} $\iff$ $\text{Bis}(G \curvearrowright X)$ is \underline{sparse}.

Remark: these classes are, however, not the same!

- This division is \underline{impossible for groups}, and
- already appeared in work of Li and Willett (2018)
Local finiteness vs. quasidiagonality

Sometimes these classes coincide, i.e., take $G \simeq X$, where $X$ is the Cantor set and $G$ discrete group, then: $\text{Bis}(G \simeq X)$ is locally finite $\iff$ $\text{Bis}(G \simeq X)$ is sparse.

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Thank you for your attention! Questions?