Noncommutative spaces at finite resolution

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Motivation: hearing the shape of a drum

Mark Kac (1966):

Riemannian (spin) geometry: \((M, g)\) is not fully determined by spectrum of \(\Delta_M (D_M)\).

- This is considerably improved by considering besides \(D_M\) also the \(C^*\)-algebra \(C(M)\) of continuous functions on \(M\) [Connes 1989]
- In fact, the Riemannian distance function on \(M\) is equal to

\[
d(x, y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : \|[D, f]\| \leq 1\}
\]
Noncommutative geometry

So, if combined with the $C^*$-algebra $C(M)$, then the answer to Kac' question is affirmative.

Connes' reconstruction theorem [2008]:

$$(C(M), L^2(S_M), D_M) \leftrightarrow (M, g)$$
Spectral data

- The mathematical reformulation of geometry in terms of spectral data requires the knowledge of all eigenvalues of the Dirac operator.
- From a physical standpoint this is not very realistic: detectors have limited energy ranges and resolution.

We develop the mathematical formalism for (noncommutative) geometry with only part of the spectrum and/or with finite resolution.

This is in line with [D’Andrea–Lizzi–Martinetti 2014], [Glaser–Stern 2019] and based on [Connes–vS] (CMP, ongoing)
The “usual” story: spectral triples

- a $\mathbb{C}^*$-algebra $A$
- a self-adjoint operator $D$ with (local) compact resolvent and bounded commutators $[D, a]$ for $a \in A \subset A$
- both acting (boundedly, resp. unboundedly) on Hilbert space $\mathcal{H}$

Generalized distance function:
- **States** are positive linear functionals $\phi : A \rightarrow \mathbb{C}$ of norm 1
- **Pure states** are extreme points of state space
- **Distance function** on state space of $A$:

$$d(\phi, \psi) = \sup_{a \in A} \{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}$$
Towards operator systems..

(I) Given \((A, \mathcal{H}, D)\) we project onto part of the spectrum of \(D\):
- \(\mathcal{H} \mapsto PH\), projection onto closed Hilbert subspace
- \(D \mapsto PDP\), still a self-adjoint operator
- \(A \mapsto PAP\), this is not an algebra any more (unless \(P \in A\))

Instead, \(PAP\) is an operator system: \((PAP)^* = Pa^*P\).

(II) Another approach would be to consider metric spaces up to a finite resolution:
- Consider integral operators associated to the tolerance relation \(R_\epsilon\) given by \(d(x, y) < \epsilon\)

So first, some background on operator systems.
Operator systems

**Definition (Choi-Effros 1977)**

An operator system is a $\ast$-closed vector space $E$ of bounded operators. **Unital:** it contains the identity operator.

- $E$ is ordered: cone $E_+ \subseteq E$ of positive operators, in the sense that $T \in E_+$ iff
  $$\langle \psi, T \psi \rangle \geq 0; \quad (\psi \in \mathcal{H}).$$
- In fact, $E$ is matrix ordered: cones $M_n(E)_+ \subseteq M_n(E)$ of positive operators on $\mathcal{H}^n$ for any $n$.

Maps between operator systems $E, F$ are completely positive maps in the sense that their extensions $M_n(E) \rightarrow M_n(F)$ are positive for all $n$. Isomorphisms are complete order isomorphisms.
**C***-envelope of a unital operator system


A C*-extension $\kappa : E \to A$ of a unital operator system $E$ is given by a complete order isomorphism onto $\kappa(E) \subseteq A$ such that $C^*(\kappa(E)) = A$. A C*-envelope of a unital operator system is a C*-extension $\kappa : E \to A$ with the following universal property:

![Diagram](attachment:image.png)
Shilov boundaries

There is a useful description of $C^*$-envelopes using Shilov ideals.

**Definition**

Let $\kappa : E \to A$ be a $C^*$-extension of an operator system. A **boundary ideal** is given by a closed 2-sided ideal $I \subseteq A$ such that the quotient map $q : A \to A/I$ is completely isometric on $\kappa(E) \subseteq A$.

The **Shilov boundary ideal** is the largest of such boundary ideals.

**Proposition**

Let $\kappa : E \to A$ be a $C^*$-extension. Then there exists a Shilov boundary ideal $J$ and $C^*_{\text{env}}(E) \cong A/J$.

As an example consider the operator system of continuous harmonic functions $C_{\text{harm}}(\overline{D})$ on the closed disc. Then by the maximum modulus principle the Shilov boundary is $S^1$. Accordingly, its $C^*$-envelope is $C(S^1)$. 
Propagation number of an operator system

One lets $E^{\circ n}$ be the norm closure of the linear span of products of \( \leq n \) elements of $E$.

**Definition**

The propagation number $\text{prop}(E)$ of $E$ is defined as the smallest integer $n$ such that $\kappa(E)^{\circ n} \subseteq C^*_{\text{env}}(E)$ is a $C^*$-algebra.

Returning to harmonic functions in the disk we have $\text{prop}(C_{\text{harm}}(\mathbb{D})) = 1$.

**Proposition**

The propagation number is invariant under complete order isomorphisms, as well as under stable=Morita equivalence [EKT, 2019]:

$$\text{prop}(E) = \text{prop}(E \otimes_{\text{min}} \mathcal{K})$$

More generally [Koot, 2021], we have

$$\text{prop}(E \otimes_{\text{min}} F) = \max\{\text{prop}(E), \text{prop}(F)\}$$
State spaces of operator systems

- The existence of a cone $E_+ \subseteq E$ of positive elements allows to speak of states on $E$ as positive linear functionals of norm 1.
- In the finite-dimensional case, the dual $E^d$ of a unital operator system is a unital operator system with

$$E^d_+ = \{ \phi \in E^d : \phi(T) \geq 0, \forall T \in E_+ \}$$

and similarly for the matrix order.
- Also, we have $(E^d)_+^d \cong E_+$ as cones in $(E^d)^d \cong E$.
- It follows that we have the following useful correspondence:
  
  pure states on $E \leftrightarrow$ extreme rays in $(E^d)_+$
  
  and the other way around.

In the infinite-dimensional/non-unital case, this is more subtle (more later..).
Spectral truncation of the circle: Toeplitz matrices

Toeplitz operator system: truncation of $C(S^1)$ on $n$ Fourier modes

$$C(S^1)^{(n)} : \begin{pmatrix} t_{k-l} \end{pmatrix}_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \cdots & \cdots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

We have: $C_{\text{env}}^*(C(S^1)^{(n)}) \cong M_n(\mathbb{C})$ and $\text{prop}(C(S^1)^{(n)}) = 2$ (for any $n$).

One can show [vS 2020, Hekkelman 2021] that state spaces on $C(S^1)^{(n)}$ (with Connes’ distance) Gromov–Hausdorff converge to $S(C(S^1))$. 
Dual operator system: Fejér–Riesz

We introduce the Fejér–Riesz operator system $C^*(\mathbb{Z})_n$:

- functions on $S^1$ with a finite number of non-zero Fourier coefficients:

  $$a = (\ldots, 0, a_{-n+1}, a_{-n+2}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{n-2}, a_{n-1}, 0, \ldots)$$

- an element $a$ is positive iff $\sum_k a_k e^{ikx}$ is a positive function on $S^1$.

The Shilov boundary of the operator system $C^*(\mathbb{Z})_n$ is $S^1$. Consequently, the $C^*$-envelope of $C^*(\mathbb{Z})_n$ is given by $C^*(\mathbb{Z})$.

**Proposition**

1. **The extreme rays in** $(C^*(\mathbb{Z})_n)_+$ **are given by the elements** $a = (a_k)$ **for which the Laurent series** $\sum_k a_k z^k$ **has all its zeroes on** $S^1$.

2. **The pure states of** $C^*(\mathbb{Z})_n$ **are given by** $a \mapsto \sum_k a_k \lambda^k$ ($\lambda \in S^1$).
Pure states on the Toeplitz matrices

Duality of $C(S^1)^{(n)}$ and $C^*(\mathbb{Z})_{(n)}$ [Connes–vS 2020] and [Farenick 2021]:

$$C(S^1)^{(n)} \times C^*(\mathbb{Z})_{(n)} \to \mathbb{C}$$

$$\left( T = (t_{k-l})_{k,l}, a = (a_k) \right) \mapsto \sum_k a_k t_{-k}$$

**Proposition**

1. The extreme rays in $C(S^1)^{(n)}$ are $\gamma(\lambda) = |f_\lambda\rangle \langle f_\lambda|$ for any $\lambda \in S^1$.
2. The pure state space $\mathcal{P}(C(S^1)^{(n+1)}) \cong \mathbb{T}^n / S_n$. 

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Spectral truncations of the circle \((n = 3)\)

We consider \(n = 3\) for which the Toeplitz matrices are of the form

\[
T = \begin{pmatrix}
t_0 & t_{-1} & t_{-2} \\
t_1 & t_0 & t_{-1} \\
t_2 & t_1 & t_0 \\
\end{pmatrix}
\]

The pure state space is \(\mathbb{T}^2/S_2\), given by vector states \(|\xi\rangle \langle \xi|\) with

\[
\xi \propto \begin{pmatrix} 1 \\ e^{ix} + e^{iy} \\ e^{i(x+y)} \end{pmatrix}
\]

This is a Möbius strip!
More on non-unital operator systems

Consider a matrix-ordered operator space \((E, \| \cdot \|)\).

- The noncommutative (nc) state space is defined for any \(n\) as
  \[
  S_n(E) := \{ \phi \in M_n(E)^*, \|\phi\| = 1, \phi \geq 0 \}
  \]
  not always convex nor weakly \(*\)-compact

- The nc quasi-state space is defined for any \(n\) as
  \[
  \tilde{S}_n(E) := \{ \phi \in M_n(E)^*, \|\phi\| \leq 1, \phi \geq 0 \}
  \]
  convex and weakly \(*\)-compact

- The modified numerical radius \(\nu_E: M_n(E) \rightarrow \mathbb{C}\) is defined as
  \[
  \nu_E(x) = \sup \left\{ |\phi \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix}| : \phi \in \tilde{S}_{2n}(E) \right\}.
  \]

**Definition (Werner 2002)**

A non-unital operator system is given by a matrix-ordered operator space for which \(\nu_E(\cdot) = \| \cdot \|\).
Approximate order units

We now consider a particular class of non-unital operator systems.

**Definition (Ng 1969)**

Let \( E \) be a matrix-ordered \(*\)-vector space. An approximate order unit for \( E \) is an ordered net \( \{e_\lambda\}_{\lambda \in \Lambda} \) of positive elements such that

for each \( x^* = x \in E \) there exists a positive real number \( t \) and \( \lambda \in \Lambda \) such that

\[-te_\lambda \leq x \leq te_\lambda.\]

In fact, if the approximate order unit is matrix-norm-defining in the sense that

\[\|x\| = \inf \left\{ t : \begin{pmatrix} te_\lambda^n & x \\ x^* & te_\lambda^n \end{pmatrix} \in M_{2n}(E)_+ \text{ for some } \lambda \in \Lambda \right\}\]

then \( E \) is a non-unital operator system [Karn 2005, Han 2010].
Assuming the existence of a **matrix-norm-defining approximate order unit** in $E$ we may show familiar $C^*$-results such as

1. the nc state space $S_n(E)$ is convex

and if $E \subseteq A$ with a norm-defining approximate order unit for $A$ contained in $E$ we have that

2. any (pure) nc state on $E$ can be extended to a (pure) state on $A$.

3. **(Jordan decomposition)** For each hermitian continuous linear functional $\phi : M_n(E) \to \mathbb{C}$ there exist positive linear functionals $\phi_+, \phi_- : M_n(E) \to \mathbb{C}$ such that $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$

4. we have an isometrical order isomorphism

$$M_n(A)^*_h/M_n(E)^{\perp}_h \to M_n(E)^*_h$$

This also applies if we replace $E$ and $A$ by dense subspaces $\mathcal{E}$ and $\mathcal{A}$. 
Operator systems, groupoids and bonds

Recall:

- Consider a locally compact groupoid $G$ equipped with a (left invariant) Haar system $\nu = \{\nu_x\}$.
- The space $C_c(G)$ of compactly supported complex-valued continuous functions on $G$ becomes a $\ast$-algebra with the convolution product and involution given by

$$f \ast g(\gamma) = \int_{G_x} f(\gamma \gamma_1^{-1}) g(\gamma_1) d\nu_x(\gamma_1); \quad f^\ast(\gamma) = \overline{f(\gamma^{-1})},$$

where $x = s(\gamma)$ for any $\gamma \in G$.
- $C_c(G)$ can be completed to the groupoid $C^\ast$-algebra $C^\ast(G)$.
**Definition**
A bond is a triple \((G, \nu, \Omega)\) consisting of a locally compact groupoid \(G\), a Haar system \(\nu = \{\nu_x\}\) and an open symmetric subset \(\Omega \subseteq G\) containing the units \(G^{(0)}\).

**Proposition**
Let \((\Omega, G, \nu)\) be a bond. The closure of the subspace \(C_c(\Omega) \subseteq C_c(G)\) in the \(C^*\)-algebra \(C^*(G)\) is an operator system.

**Example**
1. Consider \(\Omega_n = (-n, n) \subseteq \mathbb{Z} \leadsto \text{Fejér–Riesz operator system inside } C^*(\mathbb{Z})\).
2. Consider \(\Omega_n = (-n, n) \subseteq C_m\) (so modulo \(m\)). The operator system consists of banded \(m \times m\) circulant matrices of band width \(n\). Thus, the ambient groupoid is crucial since these two operator systems are not even Morita equivalent.
3. Given the set \(X = \{1, \ldots, m\}\) consider a “band” \(R_n \subseteq X \times X\) around the diagonal of width \(n \leadsto \text{banded } m \times m\) matrices of band width \(n\).
Operator systems associated to tolerance relations

- Suppose that $X$ is a set and consider a relation $\mathcal{R} \subseteq X \times X$ on $X$ that is reflexive, symmetric but not necessarily transitive.
- **Key motivating example:** a metric space $(X, d)$ with the relation $\mathcal{R}_\epsilon := \{(x, y) \in X \times X : d(x, y) < \epsilon\}$
- If $(X, \mu)$ is a measure space and $\mathcal{R} \subseteq X \times X$ an open subset we obtain the operator system $E(\mathcal{R})$. Note that $E(\mathcal{R}) \subseteq \mathcal{K}(L^2(X))$

**Example**

Let $X$ be a finite set and $\mathcal{R} \subseteq X \times X$ a symmetric reflexive relation on $X$ and suppose that $\mathcal{R}$ generates the full equivalence class $X \times X$ (i.e. the graph corresponding to $\mathcal{R}$ is connected). Then

1. the $C^*$-envelope of $E(\mathcal{R})$ is $\mathcal{K}(\ell^2(X))$ and $\text{prop}(E(\mathcal{R})) = \text{diam}(\mathcal{R})$.
2. the pure states of $E(\mathcal{R})$ are given by vector states $|v\rangle \langle v|$ for the support of $v \in \ell^2(X)$ is $\mathcal{R}$-connected.
Finite partial partitions of a metric measure space

A finite partial $\epsilon$-partition of $X$ is a finite collection $P = \{U_i\}$ of disjoint measurable sets $U_i \subseteq X$ such that $\text{diam}(U_i) < \epsilon$; directed by refinement.

- The corresponding finite-dimensional algebra $A_P$ with unit $e_P$ is
  \[
  A_P = \left\{ \sum_{U,V \in P} a_{UV}|1_U \rangle \langle 1_V| : a_{UV} \in \mathbb{C} \right\} \cong \mathcal{K}(l^2(P))
  \]

- A tolerance relation $\mathcal{R}^P_{\epsilon}$ on the finite set $P$ is given by
  \[
  \mathcal{R}^P_{\epsilon} = \{ U \times V \mid U, V \in P, \ U \times V \subseteq \mathcal{R}_{\epsilon} \} \subseteq P \times P
  \]
  and yields the operator system $E(\mathcal{R}^P_{\epsilon})$.

- If $P \leq P'$ then $E(\mathcal{R}^P_{\epsilon}) \subseteq E(\mathcal{R}^{P'}_{\epsilon})$ and also $A_P \subseteq A_{P'}$.

- Approximate order unit $\{e_P\}_P$ of $\lim \to A_P$ is contained in $\lim \to E(\mathcal{R}^P_{\epsilon})$.
Spaces at finite resolution

**Proposition**
Let $X$ be a path metric measure space with a measure of full support.

1. $\mathcal{E}(\mathcal{R}_\epsilon) := \lim E(\mathcal{R}_\epsilon^P)$ is a dense subspace of $E(\mathcal{R}_\epsilon)$
2. $\mathcal{A}_\epsilon := \lim \mathcal{A}_P$ is a dense $\ast$-subalgebra of the $C^*$-algebra $\mathcal{K}(L^2(X))$;
3. there exists a matrix-norm-defining approximate order unit for $\mathcal{A}_\epsilon$ which is contained in $\mathcal{E}(\mathcal{R}_\epsilon)$.

**Proposition**
Let $X$ be a complete, locally compact path metric measure space with a measure of full support. Then

1. $C^*_{\text{env}}(E(\mathcal{R}_\epsilon)) = \mathcal{K}(L^2(X))$.
2. The pure states of $E(\mathcal{R}_\epsilon)$ are given by vector states $|\psi\rangle\langle\psi|$ where the essential support of $\psi \in L^2(X)$ is $\epsilon$-connected.

Thanks!