Mathematics of the Dilute Bose gas

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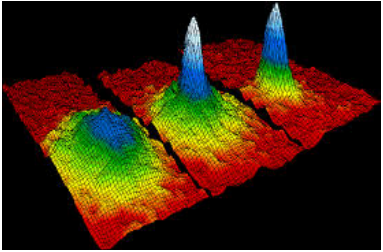
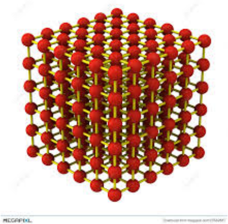
Bose-Einstein condensation

The Nobel Prize in Physics of 2001 was awarded to Eric A. Cornell, Wolfgang Ketterle and Carl E. Wieman *for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates*.



Proving BEC is difficult

Phase transition $\sim$ long range order. Similar problems concerning crystalization and magnetization are very difficult.



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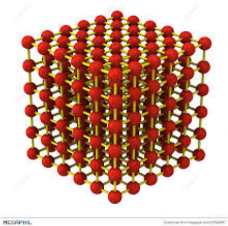

Instead, we try to understand the ground state energy (the lowest eigenvalue of the Hamiltonian). This is hard but within reach.

The most important result in linear algebra never taught

We will rephrase the spectral problem as a variational problem.

Theorem (Min-max principle)

Let $A = A^*$ be a self-adjoint $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$.
Then λ_1 satisfies

$$\lambda_1 = \inf_{v \in \mathbb{C}^n \setminus \{0\}} \frac{\langle v, Av \rangle}{\|v\|^2}.$$

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Moreover,

$$\lambda_j = \inf_{\{V \subset \mathbb{C}^n, \dim V = j\}} \sup_{v \in V \setminus \{0\}} \frac{\langle v, Av \rangle}{\|v\|^2}.$$

The dilute Bose gas

Consider N interacting, non-relativistic bosons in a box $\Lambda := [-L/2, L/2]^3$.
Let $N \in \mathbb{N}$, $\rho := N/|\Lambda| = N/L^3$.
The Hamiltonian of the system is

$$H_N := \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j),$$

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The energy density in the thermodynamic limit is

$$e(\rho) = \lim_{L\to\infty, N/|\Lambda|=\rho} E_0(N, \Lambda)/L^3.$$

The scattering length

The scattering equation,

$$(-\Delta + \frac{1}{2}v(x))(1-\omega(x)) = 0, \qquad \text{with } \omega \to 0, \text{ as } |x| \to \infty.$$

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We introduce the function

$$g := v(1-\omega).$$

Then, the scattering equation can be reformulated as

$$-\Delta\omega = \frac{1}{2}g, \qquad \text{i.e. } \widehat{\omega}(k) = \frac{\hat{g}(k)}{2k^2}.$$

Also,

$$a = (8\pi)^{-1}\int g, \qquad a_0 := (8\pi)^{-1}\int v.$$

The two-term formula

We study $e(\rho)$ in the dilute limit $\rho \to 0$. Expected:

$$e(\rho) = 4\pi\rho^2 a(1 + \frac{128}{15\sqrt{\pi}}(\rho a^3)^{1/2}) + \rho^2 a\, o((\rho a^3)^{1/2}).$$

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- Lenz (1929), Bogoliubov (1947), Lee-Huang-Yang (1957).
- Rigorous proof of leading term Dyson (1957, upper), Lieb-Yngvason (1998).
- Upper bounds giving second order term: Erdös-Schlein-Yau (2008), Yau-Yin (2009).
- Study of the limit for v becoming 'soft' as $\rho \to 0$: Lieb-Solovej, Giuliani-Seiringer (2008), Brietzke-Solovej (2019).
- Bogoliubov theory for confined Bose gases Boccato-Brennecke-Cenatiempo-Schlein (2017-2018).
- Brietzke-F-Solovej (2019)