Solving the unsolvable

Associate Professor Larisa Beilina's work on solving coefficient inverse problems develops a new mathematical idea which can lead to direct practical developments. Here, she discusses some of the greatest challenges faced during her project

To begin, what are coefficient inverse problems (CIPs), and what challenges are associated with solving CIPs in more than one dimension?

A CIP involves the reconstruction of a spatially dependent coefficient of a partial differential equation (PDE) inside a domain of interest, using measurements either of the entire boundary or of part of the boundary of said domain. The unknown coefficient, or different coefficients, of a PDE usually describes a spatially distributed physical property of the medium, such as dielectric permittivity, electric conductivity, sound speed, etc. Thus, one can image the interior of the medium by computing the spatial dependence of this coefficient – we can create an image of the interior of the medium by visualising the spatial distribution of the coefficient. This is why CIPs have a broad range of applications in imaging of hidden explosives, geophysics (subsurface exploration to detect gas or oil distribution), non-destructive testing of materials, medical imaging, etc.

Because of the applied aspect, it is important to develop reliable numerical methods for the solution of multidimensional CIPs. However, this goal poses big challenges because of two combined phenomena: nonlinearity and ill-posedness of CIPs. The most important question in this regard is about obtaining a good approximation for the unknown coefficient under the condition that information about a small neighbourhood of this coefficient is unavailable.

You are part of a large collaborative project, 'Adaptive finite element methods for solutions of inverse problems' involving Chalmers University of Technology, Gothenburg University and leading universities in Russia. Can you outline the main aims of this programme?

This collaborative project includes a two-sided scientific exchange with short and long visits to Russia and Sweden, PhD student supervision and yearly workshop organisation.

The main aim of the project is to develop a new mathematical idea – the adaptivity technique – to the solution of CIPs in imaging using electromagnetic waves, as well as in signal reconstruction in scanning electron tomography. Dr Nikolay Koshev and I developed an adaptive finite element method for the solution of the Fredholm integral equation of the first kind, and verified it on experimental data in signal reconstruction in scanning electron microscopy. Two articles on this method are accepted for publication in refereed journals.

I have also organised two workshops on inverse problems in Sweden, in 2011 and in 2012, one at Chalmers and another one at Sunne together with another satellite workshop 'Large-scale computations'.

The main topic has been the presentation of new analytical developments and new numerical methods for solutions of inverse problems. These workshops have attracted attention from worldwide researchers in the field and, as a result of their importance, I obtained a contract with Springer to publish the conference proceeding in *Proceedings in Mathematics*, which will be published this year.

At what stage is your work to find new numerical methods for the solution of the time-dependent Maxwell equations?

I am the grant holder and leader of the project 'Global convergence and adaptivity for coefficient inverse problems for Maxwell equations' supported by the Swedish Research Council. One PhD student at Gothenburg University now works under my supervision to achieve the goals of this project. We have a close collaboration with the Optical Center of the University of North Carolina at Charlotte, USA, and we are now actively testing our new reconstruction methods for the solution of CIPs for Maxwell equations on experimental data provided by the Center.

You have recently published a book: *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems*, co-written with Professor Michael V Klibanov. Who is this book intended for and why did you feel now was the right time to publish a book on CIPs?

This is the first book in which two new concepts of numerical solutions of multidimensional CIPs for hyperbolic PDEs are presented: approximate global convergence and the adaptive finite element method. This book is intended for all researchers and engineers interested in the practical solution of CIPs. The first chapter can be also used as an introductory course to the theory of ill-posed problems.

Where do you see your studies moving in the next five to 10 years? Do you have plans for any new research directions?

I hope that our methods will be applied not only for the reconstruction of dielectrics of explosives, but also in medical imaging (in diagnostics of malignant tumours of millimetre-size inclusions) and in geological profiling (in detection of gas and oil structures).

The value of convergence

Quantitative material reconstruction is the objective of a theoretical and practical project by researchers from **Chalmers University of Technology** and **Gothenburg University**, which is opening up new possibilities for the development of a wide range of applications, from medical imaging to military radar

THE ABILITY TO analyse the internal composition of an unknown object in an unobtrusive manner while keeping at a safe distance from it is, naturally, one that would offer a wide range of practical applications: from airport security to medical diagnosis, up to the detection of oil or gas under layers of rock. Many fields would benefit from such a technology.

This is the kind of issue coefficient inverse problems (CIPs) are intended to solve. CIPs aim to reconstruct a spatially dependent coefficient of a partial differential equation (PDE) inside a domain by only measuring its boundary or part of it – in simple words, 'seeing through' the medium to imagine its internal structure. Doing so is very complicated, particularly as any resulting value has to be accurate to be of any relevance. If applied in medical diagnosis, for example, a satisfactory result would not be one producing an image of tumour mass, but one giving information about its coefficient in sufficient quantity to prevent unnecessary treatment of benign tumours.

MULTIDIMENSIONAL PROBLEMS

Multidimensional CIPs (MCIPs) present several challenges, an issue that prevented the development of reliable numerical methods to solve them despite the topic being studied for more than half a century. Three factors combine to make these problems difficult: nonlinearity, the ill-posedness of MCIPs and, in most practical circumstances, the minimal amount of data available – for example, through a single measurement, as would be the case with an airport scanner in which the signal is generated from just one source.

A further complicating issue in a real-life setting is not having any information about the object under analysis and, as such, no data about the value of the sought coefficient. Any solution to these MCIPs would need to overcome this set of difficulties in order to provide answers leading to reliable applications in the everyday world.

TWO-STAGE PATH TO RESULTS

Associate Professor Larisa Beilina's research in the Department of Mathematical Sciences at Chalmers University of Technology and Gothenburg University in Sweden aims to overcome these problems, leading to solutions that are both sound on a mathematical level and that can be applied to real life. In 2008, working with Professor Michael V Klibanov of the University of North Carolina at Charlotte (UNC Charlotte), USA, Beilina developed an approximate globally convergent method which can be utilised for the solution of CIPs. This strategy departs from locally convergent methods – and, as such, bypasses the need to have a suitable starting point for the coefficient solution.

The system was further refined two years later to a two-stage numerical procedure, in which the aptly-named approximate globally convergent method is first used to find an approximate solution to the CIP before this is refined through

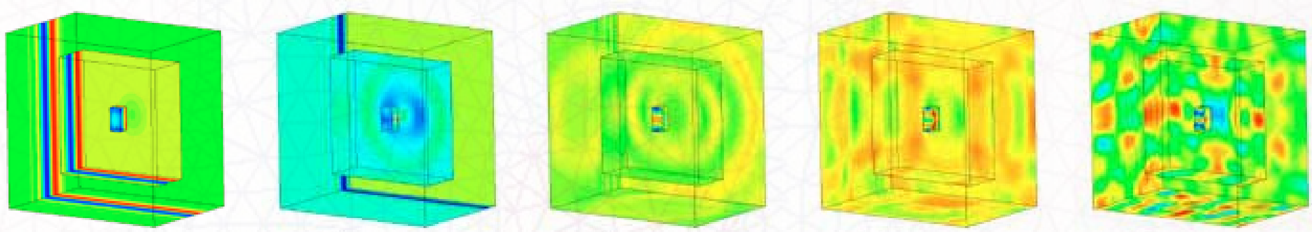

FIGURE 1. Isosurfaces of the computed solution of the acoustic wave equation reflecting from a piece of oak at different times. Software package WavES is used for the numerical simulation of this solution.

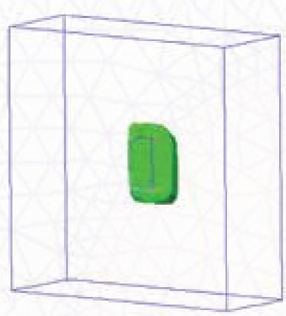

FIGURE 2. Reconstruction for a piece of oak taken from backscattered data presented in figure 1.

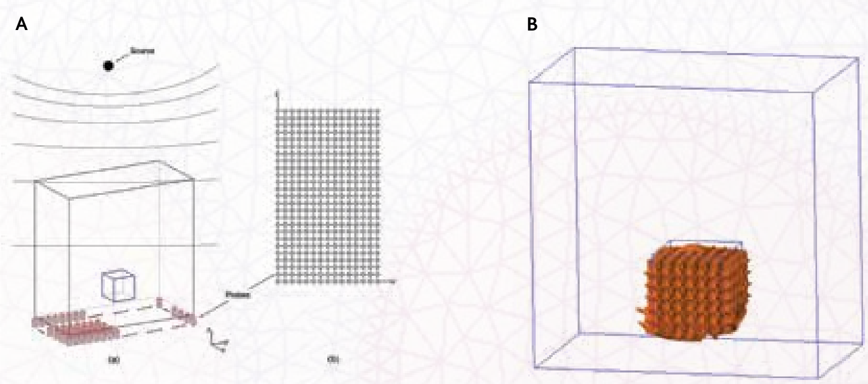

FIGURE 3. **A**. Experimental setup. The rectangular prism depicts the computational domain. Only a single source location outside of this prism was used. Tomographic measurements of the scattered time resolved electromagnetic wave were conducted on the bottom side of this prism. The signal was measured with a time interval of 20 ps with total time 12.3 ns. **B**. The final reconstruction result after applying the adaptive stage (second stage). Both the refractive index and the shape are reconstructed with excellent accuracy. Source: Beilina L and Klibanov M V, Reconstruction of dielectrics from experimental data via a hybrid globally convergent/adaptive algorithm, 2010, *Inverse problems*, **26**, 125009. Reprinted with permission.

INTELLIGENCE

QUANTITATIVE IMAGING TECHNIQUES USING ADAPTIVE AND GLOBALLY CONVERGENT ALGORITHMS

OBJECTIVES

To simultaneously: refine the approximate globally convergent numerical method for the solution of mutlidimensional coefficient inverse problems with the adaptive finite element method, that finds a point in the near vicinity of the exact solution, without any prior knowledge of that neighbourhood; ensure good performance of this method on computationally simulated and experimental data.

KEY COLLABORATORS

Professor Michael V Klibanov, Department of Mathematics and Statistics, University of North Carolina at Charlotte, USA

Professor Michael Fiddy, Optical Center, University of North Carolina at Charlotte, USA

Dr Anders Sullivan; **Dr Lam Nguyen**, US Army Research Laboratory, Adelphi, Maryland, USA

Dr Andrey V Kuzhuget; **Dr Nikolay Koshev**, private companies

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ASSOCIATE PROFESSOR LARISA BEILINA received her PhD degree in applied mathematics from Chalmers University of Technology in 2003. She then moved to Basel University for the period 2003-05 and the Norwegian University of Science and Technology in 2007-08 for postdoctoral fellowships, before returning to Sweden in the Department of Mathematical Sciences of Chalmers University of Technology and also Gothenburg University.

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an adaptive finite element method. In addition, it is worth noting that all calculations originate from the single measurement mentioned above and therefore with the least possible amount of data available.

This revised procedure achieves the two main goals that underline the research: devising a numerical method that finds a point in the near vicinity of the solution, without knowing the extent of that vicinity in advance, and achieving good performance on both computationally simulated and experimental data. Most importantly for the research, these two goals are achieved simultaneously – a result that no other numerical method for MCIPs can match, at least without unjustifiable approximations.

US ARMY VERIFICATION

Verification of this method was brought to Beilina through an experimental process conducted with the US Army Research Laboratory (ARL). The collaboration started in February 2010 when Professors Michael A Fiddy and Klibanov debated with Drs Anders Sullivan and Lam Nguyen of the ARL about the possibility of the globally convergent method being applied to the ARL's forward-looking radar.

This instrument is supposed to detect and possibly identify explosive devices lying on the ground or buried a few centimetres below the surface – a common circumstance with many improvised explosive devices utilised by insurgents in Afghanistan. The method devised by Beilina and Klibanov could help in imaging dielectric constants of such targets, thus providing an important additional piece of information and refining the criteria for discrimination of targets of the forward-looking radar, which normally simply relies on the intensity of the radar image.

TESTING IN THE FIELD

The Beilina-Klibanov method was subjected to tests by the US Army Research Office that were deliberately challenging and aimed at replicating the same amount of information, or lack of, to be found in a real-life situation. No data was given to the team about the substance of the targets, just an indication of whether they were buried or on the ground. In addition to the blind data, the level of difficulty was increased by the heterogeneous nature of the targets themselves and the fact that they were placed in a cluttered environment, one which would be expected in a real-life scenario.

The data produced by the radar was one-dimensional – an ulterior difficulty, given that the theory required three-dimensional data for the three-dimensional targets analysed. It was clear from the start that the only result the method could provide from these starting points would be the values of dielectric constants, but the results were surprisingly good. Values produced were very accurate and all such constants were within well-tabulated limits.

These field tests, and further studies, all found their way into new scientific software devised by Beilina called WavES (Wave Equations Solutions). WavES is a combined theoretical and practical tool for the high-performance numerical solution of time-dependent wave equations (acoustic, elastic and electromagnetic). The software is used for implementing an approximate globally convergent method and adaptive finite element method to solve MCIPs. Essentially, WavES is a C++ software package that can be used to solve time-dependent wave equations.

FROM BATTLEFIELD TO SURGERY

The project encompasses both theoretical and practical elements, and implements the approximate globally convergent algorithm; in particular, this solution is currently used by the Optical Center of UNC Charlotte for qualitative reconstruction of dielectrics in imaging of explosives. The Center is putting the method into practice to produce a prototype of real-life working equipment; tests to create such a device are taking place in collaboration with ARL.

The application of this mathematical method to real-world scenarios is one of the main aims for Beilina, who hopes the scope of her research will extend beyond the realm of military technology and into other valuable sectors such as medicine and even the energy industry; for example, profiling geology in search for natural resources is a possible use. "I am very lucky I don't need to wait 20 years to see my theoretical method implemented in real-life applications," she enthuses.

CONVERGING DISCIPLINES

With her mind now focused on the development of new and improved methods for the solution of CIPs – her new mathematical idea, the adaptive finite element method, is being constantly refined – Beilina is the leader of a new project entitled 'Global convergence and adaptivity for coefficient inverse problems for Maxwell equations'. As Maxwell's equations are ubiquitous in real-world applied physics, there is little doubt that this new direction will spawn further practical applications.

"Mathematicians are often scared of physicists testing their new methods, and physicists are often dismissive of new ideas from mathematicians," Beilina exclaims. As in her previous efforts, the highlight of her work will be in bringing the two groups together to create something more than the sum of its parts.