Piecewise linear interpolation of noise in finite element approximations of parabolic SPDEs

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**Stochastic partial differential equation (SPDE) setting**

Consider a stochastic advection-reaction-diffusion equation in $[0, T] \times \mathcal{D}$,

$$
\frac{\partial X}{\partial t}(t, x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left( a_{i,j} \frac{\partial X}{\partial x_i} \right)(t, x) + b(x) \cdot \nabla X(t, x) + f(X(t, x), x) + g(X(t, x), x) \frac{\partial W(t)}{\partial t}(x)
$$

with Gaussian noise $W$ such that $\text{Cov}(W(t, x), W(s, y)) = \min(t, s)q(x, y)$. This is modeled as an Itô SDE in $H = L^2(\mathcal{D})$ (with $\mathcal{D} \subset \mathbb{R}^2$ a convex polygon),

$$
\text{d}X(t) + AX(t) = F(X(t)) \text{d}t + G(X(t)) \text{d}W(t), \quad t \in (0, T],
$$

$$
X(0) = X_0,
$$

where $A$ is elliptic with either Dirichlet or Neumann boundary conditions, $F, G$ Lipschitz non-linearities and $W$ a standard cylindrical Wiener process in the reproducing kernel Hilbert space $H_q(\mathcal{D})$.

We consider fully discrete approximations $X_{h, \Delta t}$ for such SPDEs based on piecewise linear finite elements on a triangulation $\mathcal{T}_h$ of $\mathcal{D}$ with maximal mesh size $h$ and a semi-implicit Euler scheme with time step size $\Delta t$. 

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Noise discretization

In the approximation we must, with $\Delta W^j = W(t_{j+1}) - W(t_j)$, compute

$$\mathcal{W}_k = \left\langle G(X^j_{h, \Delta t}) \Delta W^j, \phi_k \right\rangle_H = \int_D g(X^j_{h, \Delta t}(x), x) \Delta W^j(x) \phi_k(x) \, dx$$

for each time $t_{j+1}$ and nodal basis function $\phi_k$, $k = 1, \ldots, N_h$.

Conditioned on $X^j_{h, \Delta t}$, $(\mathcal{W}_k)_{k=1}^{N_h}$ is Gaussian with covariance matrix

$$\text{Cov}(\mathcal{W}_k, \mathcal{W}_\ell) = \Delta t \int_D \int_D g(X^j_{h, \Delta t}(x), x) g(X^j_{h, \Delta t}(y), y) \phi_k(x) \phi_\ell(y) q(x, y) \, dx \, dy.$$ 

We avoid computing this matrix and performing a Cholesky decomposition at each time $t_j$ by discretizing $\Delta W^j$ via piecewise linear interpolation.

What is the resulting noise discretization error?
Piecewise linear interpolation of noise

We assume that $q$ is positive semidefinite on $S \supseteq D$ with an additional triangulation $T_{h'}$ on $S$. We replace

$$\Delta W^j \rightarrow I_h R_{S \rightarrow D} I_{h'} \Delta \tilde{W}^j.$$ 

Here $I_h$ and $I_{h'}$ are piecewise linear interpolants with respect to $T_h$ and an additional triangulation $T_{h'}$ on $S$ while $R_{S \rightarrow D}$ restricts functions on $S$ to $D$ and $\tilde{W}$ is an extension of $W$ to a cylindrical Wiener process in $H_q(S)$. This is made rigorous by

(i) constructing the extension $\tilde{W}$,
(ii) realizing that evaluation functionals are continuous on RKHS and
(iii) considering $I_h R_{S \rightarrow D} I_{h'}$ as mapping one cylindrical Wiener process to another.

In terms of the basis functions $\phi^j_k$ associated to the $N_{h'}$ nodes $x'_k$ of $T_{h'}$,

$$I_{h'} \Delta \tilde{W}^j = \sum_{k=1}^{N_{h'}} \Delta \tilde{W}^j(x'_k) \phi^j_k \text{ with } \text{Cov}(\Delta \tilde{W}^j(x'_k), \Delta \tilde{W}^j(x'_\ell)) = \Delta t q(x'_k, x'_\ell).$$

(i) Need only sample $N_{h'}$ values from $q$ without calculating integrals involving $q$.
(ii) Applicable to the circulant embedding method on a square $S$ with a uniform mesh.
(iii) Straightforward to implement in modern finite element software such as FEniCS.
Stochastic reaction-diffusion eq. with multiplicative Matérn noise, $t = 0.9, 0.95, 1.0$. 
Bound on noise discretization error

1. $q$ is bounded and $\exists \gamma \in (0, 1] : q(x, x) - 2q(x, y) + q(y, y) \leq C|x - y|^{2\gamma}, \; x, y \in S$

2. For some $\mu > 0$ there is a $p > 2/\mu$ such that $H_q(S) \hookrightarrow W^{\mu, p}(S)$.

Suppose that $q$ satisfies these assumptions and that $g : \mathbb{R} \times \mathcal{D} \rightarrow \mathbb{R}$ defining $G$ is jointly Lipschitz. Then for $r \in [0, \min(\gamma, \theta))$ and $X_0 \in \text{dom}(A^{(r+1)/2})$ with

$$\theta = \begin{cases} 
1/2 & \text{under Dirichlet boundary conditions on } A, \\
1 & \text{under Neumann boundary conditions},
\end{cases}$$

and suitable advection-reaction terms $F$, we prove the following:

**Theorem**

Let $(T_h)_{h \in (0, 1]}$ on $\mathcal{D}$ be quasiuniform and let $(T'_h)_{h' \in (0, 1]}$ on $S$ be regular with $h \leq Ch'$. Then, $\forall s < \min(\mu, 2)$, $\exists C < \infty$ such that $\forall \Delta t, h \in (0, 1]$,

$$\sup_{j \in \{1, \ldots, N_{\Delta t}\}} \|X^j_{h, \Delta t} - X(t_j)\|_{L^p(\Omega, H)} \leq C(h^{r+1} + \Delta t^{1/2} + (h')^s).$$

A split into an SPDE approximation error and a noise discretization error.
Proof techniques

The approach follows [Kruse, 2014] with the addition of a noise discretization error bound. Main tools for this are **Hilbert–Schmidt** bounds on $G$ such as

**Lemma** (For $(G(u)v)(x) = g(u(x), x)v(x)$, $g$ Lipschitz)

\[
\forall r \in [0, \gamma), u \in H^r = W^{r,2}(\mathcal{D})
\]

\[
\|G(u)\|_{\mathcal{L}_2(H^q(\mathcal{D}), H^r)} \leq C(1 + \|u\|_{H^r}).
\]

combined with similar negative-norm bounds, interpolation techniques and smoothing of the semigroup in mild SPDE solutions.

We also generalize a **fractional Sobolev norm** bound on piecewise linear finite element interpolants from [Belgacem and Brenner, 2001] to the setting $p \neq 2$.

**Proposition**

*Let $1 < p < \infty$, $sp > 2$ and $r \in [0, s] \cap [0, 1 + 1/p)$. Then, there is a constant $C < \infty$ that does not depend on $h$ such that for all $v \in W^s_p(\mathcal{D})$,*

\[
\|(I - I_h)v\|_{W^r_p} \leq Ch^{\min(s-r,2)}\|v\|_{W^s_p}.
\]
No effect of noise discretization for Matérn covariance kernels

We study the error \( \sup_j \mathbb{E} \left[ \| X_{h,\Delta t}^j - X(t_j) \|_{L^2(D)}^2 \right]^{1/2} \) numerically with fixed \( \Delta t = 10^{-3} \) and \( h = 2^{-1}, \ldots, 2^{-5} \) in a Monte Carlo simulation. \( D \) is a regular dodecagon with center \((0.5, 0.5)\) and radius 0.5, and \( S \) the unit square. We let \( A = 10^{-2}(-\Delta + 1) \) and consider three examples of kernels \( q \) with different functions \( b, f \) and \( g \) and boundary conditions. \( X \) is replaced by a reference solution.

First example: a Matérn kernel with smoothness parameter \( \nu \in (0, 1) \), \( \mu - 1 = \gamma = \nu \). Interpolation of noise does not affect the convergence rate in this case.

\[
\text{(Neumann b.c., } q(|x - y|) = \sigma^2 2^{1-\nu} / \Gamma(\nu)(\sqrt{2\nu}|x - y|/\rho)^\nu K_\nu((\sqrt{2\nu}|x - y|)/\rho), \rho = 0.25, \sigma^2 = 10, X_0 = 0, b(x) = (0, 0), f(u, x) = f(u) = 10^{-1} + u/(|u| + 1), g(u, x) = g(u) = u/(|u| + 1) \text{ and } h' = h.\text{)}
\]
Domination of noise discretization error for factorizable kernels

Next we compare the cases that $q$ is either the exponential kernel

$$q(x, y) = \sigma^2 \exp(-|x - y|/\rho)$$

or the factorizable exponential kernel

$$q(x, y) = \sigma^2 \exp(-(|x_1 - y_1| + |x_2 - y_2|)/\rho).$$

In both cases $\gamma = 1/2$. But $H_q \hookrightarrow W^{\mu,p}$ with $\mu = 3/2, p = 2$ in the first case and with $\mu < 1$ and some $p > 2$ in the second.

(Dirichlet b.c., $\rho = 0.25, \sigma^2 = 10, X_0 = 0, f(u) = 10^{-1}, g(u) = 1, b(x) = 10^{-1}(1, 1)$ and $h' = h$.)
Sampling on coarse meshes

For smooth kernels $q$ and non-smooth initial data, the SPDE approximation error dominates. Let

$$q(x, y) = \begin{cases} \sigma^2(1 - |x - y|)^4(4|x - y| + 1) & \text{if } |x - y| \in [0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

so that $\gamma = 1, \mu = 5/2$ and let

$$X_0(x) = 3(- \log((x_1 - 0.5)^2 + (x_2 - 0.5)^2))^{1/3}, \quad x = (x_1, x_2) \in D,$$

which is in $D(A^{(r+1)/2})$ for $r = 0$ but not $r > 0$ under Neumann b.c. The error is bounded by $h + \Delta t^{1/2} + (h')^s, s < 2$. We compare the cases $h' \sim \sqrt{h}$ and $h' \sim h$.

(Left: The setting $h' \sim \sqrt{h}$. Right: $\sigma^2 = 10, f(u) = 10^{-1}, g(u) = 1$ and $b(x) = 0$.)
Thank you for listening!


Implementation in FEniCS at bitbucket.org/andreas-petersson/noise-interpolation-in-stochastic-pdes