Convex structure of unital quantum channels, factorizability and traces on the universal free product of matrix algebras

Magdalena Musat
University of Copenhagen

Noncommutativity in the North
Chalmers University, Gothenburg

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Central theme: A certain class of completely positive maps (factorizable maps), introduced by C. Anantharaman-Delaroche '05.

Their study **led to** investigating the convex structure of the set of unital quantum channels, interesting applications in the analysis of QIT (e.g., settling in the negative the Asymptotic Quantum Birkhoff Conjecture) and revealed infinite dim phenomena therein, connections to/reformulations of the Connes Embedding Problem, and recently, through a new view-point, some interesting problems in operator algebras.

For $n \geq 2$, consider following sets of maps $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$:

$$\mathcal{FM}(n) \subseteq \mathrm{UCPT}(n) \begin{array}{l} \subsetneq \mathrm{CPT}(n) \\ \subsetneq \mathrm{UCP}(n) \end{array}$$

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- (Choi '73): Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ linear. Then

$$T \in \mathrm{CPT}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j A_j^* = I_n.$$

$$T \in \mathrm{UCP}(n) \iff \exists A_1, \ldots, A_d \in M_n(\mathbb{C}) : Tx = \sum_{j=1}^{d} A_j^* x A_j, \ \sum_{j=1}^{d} A_j^* A_j = I_n.$$

When $\{A_1, \ldots, A_d\}$ lin independent, d is called the Choi-rank of T.

Thm (Choi '75): $T \in \partial_e(\mathrm{CPT}(n)) \iff \{A_i A_j^*\}_{i,j=1}^{d}$ lin independent.
Respectively, $T \in \partial_e(\mathrm{UCP}(n)) \iff \{A_i^* A_j\}_{i,j=1}^{d}$ lin independent.

Thm (Landau-Streater '93): $T \in \partial_e(\mathrm{UCPT}(n)) \iff \{A_i^* A_j \oplus A_j A_i^*\}_{i,j}$ lin independent.

Hence $\partial_e(\mathrm{UCPT}(n)) \supseteq (\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n)$.

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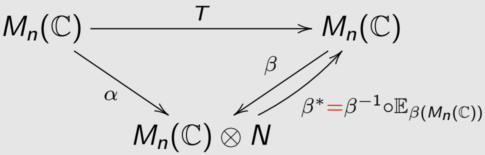

▶ α, β are injective (thus embeddings) and trace-preserving. Since unital embeddings of $M_n(\mathbb{C})$ into a vN alg are unitarily equiv, can take

$$\beta(x) = x \otimes 1_N, \quad \alpha(x) = u^* \beta(x) u, \quad x \in M_n(\mathbb{C}),$$

for some $u \in M_n(\mathbb{C}) \otimes N$ unitary; N can be taken II_1-vN alg (even factor).

Theorem (Haagerup-M '11): $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a factorizable quantum channel **iff** $\exists$ (N, τ_N) finite vN algebra (called ancilla) and a unitary $u \in M_n(\mathbb{C}) \otimes N$: $Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)$, $x \in M_n(\mathbb{C})$.

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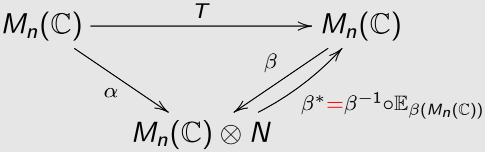

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- (R. Werner): Factorizable channels are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

- Automorphisms of $M_n(\mathbb{C})$ are factorizable.

Let $\mathcal{FM}(n)$ denote all factorizable quantum channels on $M_n(\mathbb{C})$, $n \geq 2$. Then $\mathcal{FM}(n)$ is convex and closed.

Proposition (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, with *canonical form*

$$Tx = \sum_{i=1}^{d} A_j^* x A_j, \quad x \in M_n(\mathbb{C}).$$

If $d := \text{Choi-rank}(T) \geq 2$ and $\{A_i^* A_j\}_{1 \leq i,j}^{d}$ lin indep, then $T \notin \mathcal{FM}(n)$.

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- (Kümmerer '83): $\mathrm{UCPT}(2) = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$. However,

$$\mathrm{UCPT}(n) \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))), \quad n \geq 3,$$

cf. (Kümmerer '86, Landau-Streater '93, Kümmerer-Maasen '87).

- **Asymptotic Quantum Birkhoff Conj** (Smolin-Verstraete-Winter '05): Any $T \in \mathrm{UCPT}(n)$, $n \geq 3$, satisfies

$$\lim_{k\to\infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0.$$

Theorem (Haagerup-M '11): Let $T \in \mathrm{UCPT}(n)$, $n \geq 3$. For $k \geq 1$,

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- If $T \notin \mathcal{FM}(n)$, then $d_{\mathrm{cb}}(T, \mathcal{FM}(n)) > 0$.

As $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subset \mathcal{FM}(n)$, any non-factoriz channel fails **AQBC**.

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- For $T \in \mathrm{UCPT}(n)$, Choi-rank$(T) = 1$ **iff** $T \in \mathrm{Aut}(M_n(\mathbb{C}))$.

Set $\partial_e^*(\mathrm{CPT}(n)) = \partial_e(\mathrm{CPT}(n)) \setminus \mathrm{Aut}(M_n(\mathbb{C}))$, and similarly $\partial_e^*(\mathrm{UCP}(n))$.

Cor: $T \in (\partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\mathrm{UCPT}(n)$:

- $\partial_e(\mathrm{UCPT}(n)) \setminus ((\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))) \cap \mathrm{UCPT}(n))$ can contain both factorizable and non-factorizable maps.

- $T \in \partial_e^*(\mathrm{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\mathrm{UCP}(n)) \cup \partial_e^*(\mathrm{CPT}(n))$.
- (Ohno '09): $\exists T \in \partial_e^*(\mathrm{UCPT}(3))$, Choi-rank 4; (H-M-R): $T \notin \mathcal{FM}(3)$.
- (H-M-R '21): Explicit family $T_t \in \partial_e^*(\mathrm{UCPT}(3)) \cap \mathcal{FM}(3)$, Choi-rank 4.

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Cor: $T \in (\partial_e^*(\text{UCP}(n)) \cup \partial_e^*(\text{CPT}(n))) \cap \text{UCPT}(n) \Rightarrow T \notin \mathcal{FM}(n)$.

Remark: Not easy to characterize non-factorizability in terms of the convex structure of $\text{UCPT}(n)$:

- $\partial_e(\text{UCPT}(n)) \setminus \big((\partial_e(\text{UCP}(n)) \cup \partial_e(\text{CPT}(n))) \cap \text{UCPT}(n)\big)$ can contain both factorizable and non-factorizable maps.

▶ $T \in \partial_e^*(\text{UCPT}(n))$, Choi-rank $> n \Rightarrow T \notin \partial_e^*(\text{UCP}(n)) \cup \partial_e^*(\text{CPT}(n))$.

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A class of UCPT(n) maps constructed in (Haagerup-M-Ruskai '21):

Given $n \geq 3$, $V_1, \ldots, V_n \in \mathcal{U}(n-1)$ and $t \in [-1, 1]$, $t \neq -1/(n-1)$, set

$$A_m = \frac{1}{\sqrt{n-1-t^2}} S^{-m+1} \begin{pmatrix} t & 0 \\ 0 & V_m \end{pmatrix} S^{m-1}, \quad 1 \leq m \leq n.$$

Here S is the canonical shift: $S(e_j) = e_{j+1}$, for $1 \leq k \leq n-1$ and $S(e_n) = e_1$, where $\{e_j\}_{j=1}^n$ the canonical unit vector basis in $\mathbb{C}^n$.

- Can verify $\sum_{m=1}^n A_m^* A_m = I_n = \sum_{m=1}^n A_m A_m^*$. Thus, if

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Theorem A (H-M-R '21): For $n \geq 3$ and $t \in (-1, 1)$, $t \neq -1/(n-1)$, there exists $W = W^* \in \mathcal{U}(n-1)$ such that if $V_1 = \ldots = V_n = W$ and $\{A_m\}_{m=1}^n$ are as above, then $\{A_i^* A_j\}_{i,j=1}^n$ linearly independent. Hence

$$T \in \partial_e^*(UCP(n)) \cap \partial_e^*(CPT(n)),$$

thus T is non-factorizable.

Proof: Lots of linear algebra.

Theorem B (H-M-R '21): For $n \geq 3$ and $t \in (-1, 1)$, $t \neq -1/(n-1)$, the set of n-tuples $(V_1, \ldots, V_n) \in \mathcal{U}(n-1)^n$ such that $\{A_i^* A_j\}_{i,j=1}^n$ is linearly indep has co-measure 0 w.r.t. Haar measure. Hence almost all quantum channels T arising in this way are non-factorizable.

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Infinite dimensional phenomena in QIT:

Question: Do we need (inf dim) vN algs to describe factorizable channels?

- For a factorizable channel, *minimal* ancilla (and its size) **not** unique.

E.g., consider the completely depolarizing channel S_n, $n \geq 2$

$$S_n(x) := \mathrm{tr}_n(x)1_n = \int_{\mathcal{U}(n)} u^* x u \, d\mu(u), \quad x \in \mathbb{M}_n(\mathbb{C}).$$

It's factorizable, and possible ancillas are: $\mathbb{C}^{n^2}$, $M_n(\mathbb{C})$, but also (a corner of) the reduced free product von Neumann alg $(M_n(\mathbb{C}), \mathrm{tr}_n) * (M_n(\mathbb{C}), \mathrm{tr}_n)$.

Let $\mathcal{FM}_{\text{fin}}(n)$ = factoriz channels on $M_n(\mathbb{C})$ admitting a finite dim ancilla.

Theorem (Rørdam-M '19): $\mathcal{FM}_{\text{fin}}(n)$ is not closed, whenever $n \geq 11$. Moreover, for each such n, there exist factorizable quantum channels on $M_n(\mathbb{C})$ which do require infinite dimensional (even type II_1) ancilla.

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Note: If $B \in M_n(\mathbb{C})$ is a *correlation matrix* (i.e., pos definite, all diagonal entries equal 1), then the associated Schur multiplier (= Schur prod map) $T_B \in UCPT(n)$. If B has real entries, then T_B is factorizable. (Ricard '08)

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(Rørdam-M '20): A new view-point on factorizable channels, leading to further connections (and interesting open problems in C*-algebras):

- $\mathcal{FM}(n)$ is *parametrized by* simplex of tracial states $T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$.

More precisely, if $\tau \in T(M_n(\mathbb{C}) * M_n(\mathbb{C}))$, let

$$C_\tau(i,j;k,\ell) = n\tau\big(\iota_2(e_{k\ell})^* \iota_1(e_{ij})\big), \qquad 1 \le i,j,k,\ell \le n,$$

where $\iota_1, \iota_2 \colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) * M_n(\mathbb{C})$ are the canonical inclusions. Then $C_\tau \in M_{n^2}(\mathbb{C})$ is positive, hence it is the Choi matrix of some c.p. lin map $T_\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, which turns out to be a factoriz quantum channel!

In fact, the map $\Phi \colon T(M_n(\mathbb{C}) * M_n(\mathbb{C})) \to \mathcal{FM}(n), \tau \mapsto \Phi(\tau) := T_\tau$ is an affine continuous surjection, satisfying, moreover,

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Thm (Haagerup-M '15) Connes Embedding Problem (CEP) has positive answer **iff** $\overline{\mathcal{FM}_{\text{fin}}(n)} = \mathcal{FM}(n)$, $\forall n \geq 3$.

Question: What can we say about $\overline{T_{\text{fin}}(M_n(\mathbb{C}) * M_n(\mathbb{C}))}$?

- (Exel–Loring '92): $M_n(\mathbb{C}) * M_n(\mathbb{C})$ residually finite dim. (RFD)
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In general, given $A =$ (sep) unital tracial C^*-algebra, we have inclusions:

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where $T_{\rm qd}(A) =$ quasi-diagonal traces, $T_{\rm am}(A) =$ amenable (liftable) traces, $T_{\rm hyp}(A) =$ hyperlinear traces (i.e., traces τ st $\pi_\tau(A)'' \hookrightarrow \mathcal{R}^\omega$).

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Reformulation of CEP: For all sep. unital tracial C^*-algs (A, τ), there is a unital trace- preserving *-hom $\varphi\colon A \to \prod_{n=1}^\infty M_{k_n}/I^\omega$, for some $k_n \geq 1$.

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As unital separable $\mathcal{Z}$-stable C*-alg are singly generated (Thiel-Winter), we deduce: If A is a simple unital inf dim AF-algebra, then $M_n(A)$ is a quotient of $M_n(\mathbb{C}) * M_n(\mathbb{C})$, $n \geq 3$.

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Connections with analysis of (synchronous) quantum correlations and CEP

- $\mathbb{F}(n,k) = \mathbb{Z}_k * \mathbb{Z}_k * \cdots * \mathbb{Z}_k$, n copies, $n, k \geq 2$.
- $C^*(\mathbb{F}(n,k)) = C^*(q_{j,x} \mid q_{j,x} = q_{j,x}^* = q_{j,x}^2,\ \sum_{j=1}^k q_{j,x} = 1)$.

Definition: A "correlation" $[(p(i,j \mid x,y)]$ is *synchronous* if $\forall\, 1 \leq x \leq n$, $p(i,j \mid x,x) = 0$ whenever $i \neq j$.

Theorem (Paulsen-Severini-Stalke-Todorov-Winter '16):

$$\begin{aligned} C_{qc}^s(n,k) &= \left\{ \left[\tau(q_{j,x} q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T(C^*(\mathbb{F}(n,k))) \right\} \\ C_q^s(n,k) &= \left\{ \left[\tau(q_{j,x} q_{i,y})\right]_{(i,x;j,y)} \mid \tau \in T_{\text{fin}}(C^*(\mathbb{F}(n,k))) \right\}. \end{aligned}$$

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Theorem (Kim-Paulsen-Schafhauser '17, Ozawa '12): TFAE

(1) Connes Embedding Problem has positive answer.

(2) $C^s_{qa}(n,k) = C^s_{qc}(n,k)$, $\forall n, k \geq 2$.

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I'd like to discuss a proof of (1) $\Longrightarrow$ (2), based on arguments of C. Schafhauser in a recent talk (AIM, May 2021).

By (PSSTW '16), suffices to show $\overline{T_{\rm fin}(C^*(\mathbb{F}(n,k)))} = T(C^*(\mathbb{F}(n,k)))$.

Proposition (Schafhauser): For all $n, k \geq 2$, we have

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Lemma (Folklore): Let $I \lhd M$, where $M =$ unital C^*-alg of real rank zero (e.g., M a vN algebra), and let $\pi\colon M \to M/I$ be the quotient mapping.
If $q_1, \dots, q_k \in M/I$ are projections s.t. $\sum_{j=1}^k q_j = 1$, then
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Since $C^*(\mathbb{Z}_k)$ is generated by k projections summing up to 1, each unital *-hom $\varphi\colon C^*(\mathbb{Z}_k) \to M/I$ has a lift:

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Corollary: Let $I \lhd M$, $\pi\colon M \to M/I$ as above. Then each unital *-hom $\varphi\colon C^*(\mathbb{F}(n,k)) \to M/I$ lifts to a unital *-hom:

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s.t. $\tau = \tau_\omega \circ \varphi$. The lift ψ exists by the previous corollary.

Write $\psi = (\psi_n)_{n\geq 1}$ with $\psi_n \colon C^*(\mathbb{F}(n,k)) \to M_{k_n}$ unital *-homs.

By definition of τ_ω, for all $a \in C^*(\mathbb{F}(n,k))$ we have

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