Central sequence algebras via nilpotent elements

Joint work with Dominic Enders

March 13, 2022
Definition A bounded sequence \((x_n)\) in a C*-algebra \(\mathcal{A}\) is central if
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[x_n, a] \to 0, \text{ for any } a \in \mathcal{A}.
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Separable C*-algebra setting: \(\| [x_n, a] \| \to 0\)

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Central sequences

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**Definition** A bounded sequence \((x_n)\) in a C*-algebra \(A\) is *central* if

\[[x_n, a] \to \omega 0, \text{ for any } a \in A.\]
Example Take $z_n \in Z(A), \ b_n \to 0$. Then

$$(z_n + b_n)$$

is a central sequence. Such central sequences are called trivial.
Central sequence algebras

\[ A_\infty = \{(x_n) \mid x_n \in A, \sup \|x_n\| < \infty\} \]
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**Definition** Central sequence algebra \( F(A) \) of a unital C*-algebra \( A \) is

\[ F(A) := A' \bigcap A_\omega. \]
If \( A \) is non-unital, \( A' \cap A_\omega \) is too big.
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\[
\text{Ann}(A, A_\omega) := \{ \pi((x_n)) \mid \|x_na\| \to 0, \|ax_n\| \to 0, \text{ for any } a \in A \}
\]
Central sequence algebras: non-unital case

If $A$ is non-unital, $A' \cap A_\omega$ is too big.

$$Ann(A, A_\omega) := \{ \pi((x_n)) \mid \|x_n a\| \to 0, \|a x_n\| \to 0, \text{ for any } a \in A\}$$

E.g. let $(e_n), (e'_n)$ be two approximate units. Then

$$\pi((e_n - e'_n)) \in Ann(A, A_\omega).$$
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E.g. let $(e_n), (e'_n)$ be two approximate units. Then

\[
\pi((e_n - e'_n)) \in Ann(A, A_\omega).
\]

**Definition** For a non-unital $C^*$-algebra $A$, its *central sequence algebra* $F(A)$ is

\[
F(A) := \left( A' \cap A_\omega \right) / Ann(A, A_\omega).
\]
Let $A$ be a $II_1$-factor.
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$F(A)$ is trivial $\iff$ $A$ has no property $\Gamma$
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J. Phillips 1988: If $A$ is unital and either simple or $A \supset K(H)$, then $F(A)$ is not abelian.
Kirchberg 2006 ...
Central sequence algebras in the classification program

Kirchberg 2006 ...

Properties of $A \iff$ properties of $F(A)$
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E.g. a separable nuclear $A$ is simple purely infinite $\iff$ $F(A)$ is simple and $F(A) \not\cong \mathbb{C}$. 
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Properties of $A \iff$ properties of $F(A)$

E.g. a separable nuclear $A$ is simple purely infinite $\iff F(A)$ is simple and $F(A) \not\cong \mathbb{C}$. 

E.g. a unital separable $A$ is $\mathbb{Z}$-absorbing \((A \cong A \otimes \mathbb{Z})\) $\iff \mathbb{Z} \hookrightarrow F(A)$.
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**Theorem (Ando-Kirchberg 2014)**

If $A$ is not type I, then $F(A)$ is not abelian.
$l_1$-factors | separable C*-algebras
--- | ---
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Question: When is $F(A)$ abelian?
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J. Phillips 1988: If $A$ is unital and either simple or $A \supset K(H)$, then $F(A)$ is not abelian.

Theorem (Ando-Kirchberg 2014)
If $A$ is not type I, then $F(A)$ is not abelian (not subhomogeneous).

Ozawa 2014: different proof in unital case.
Question: When is $F(A)$ abelian? subhomogeneous?
**Question:** When is $F(A)$ abelian? subhomogeneous?

Can assume $A$ is type I.
Definition A C*-algebra $A$ is type I (or GCR) if for any $\pi \in \hat{A}$, 
\[ \pi(A) \supseteq K(H). \]
Definition A $C^*$-algebra $A$ is *type I* (or *GCR*) if for any $\pi \in \hat{A}$,

$$\pi(A) \supseteq K(H).$$

Definition A $C^*$-algebra $A$ is *CCR* if for any $\pi \in \hat{A}$,

$$\pi(A) = K(H).$$
**Step 1:** \( A \) is type I but not CCR \( \Rightarrow F(A) \) is not abelian/subhomogeneous.
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Sufficient: to prove that $A \supset K(H) \Rightarrow F(A)$ is not abelian/subhomogeneous.
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Phillips 88: if $A \supset K(H)$ is unital, then $F(A)$ is not abelian

One needs some other technique for the non-unital case and to show non-subhomogeneity
An element $x \in A$ is *nilpotent of order* $n$ if $x^n = 0$. 

Theorem (Olsen-Pedersen 1989) \[ \text{Nilpotents are liftable: suppose } x \in A/I \text{ with } x^n = 0, \text{ then } x \text{ lifts to } a \in A \text{ with } a^n = 0. \]

Theorem (Sh. 2008) \[ \text{Nilpotent contractions are liftable.} \]

Corollary \[ \text{Given } n \in \mathbb{N} \text{ and } \epsilon > 0, \text{ there exists } \delta \text{ such that the following holds: for any } C^*-\text{algebra } A \text{ and any } x \in A \text{ satisfying } \|x^n\| \leq \delta \text{ and } \|x\| \leq 1 \text{ there is } y \in A \text{ such that } y^n = 0, \|y\| \leq 1 \text{ and } \|y - x\| \leq \epsilon. \]
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**Theorem (Sh. 2008)**

Nilpotent contractions are liftable.

**Corollary**

Given $n \in \mathbb{N}$ and $\epsilon > 0$, there exists $\delta$ such that the following holds: for any C*-algebra $A$ and any $x \in A$ satisfying $\|x^n\| \leq \delta$ and $\|x\| \leq 1$ there is $y \in A$ such that $y^n = 0$, $\|y\| \leq 1$ and $\|y - x\| \leq \epsilon$. 

Joint work with Dominic Enders
Folklore:

A C*-algebra is commutative if and only if it does not contain any non-trivial nilpotent elements.
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A C*-algebra is commutative if and only if it does not contain any non-trivial nilpotent elements.

Theorem (Hadwin 1997)
A C*-algebra $A$ is $n$-subhomogeneous if and only if each nilpotent element in $A$ has order not larger than $n$. 
Theorem (V. Shulman-Y. Turovsky 2014)

The following implications hold:

(i) $A$ is type I;

(ii) The spectral radius function $a \mapsto \rho(a)$ is continuous on $A$;

(iii) The closure of nilpotents in $A$ consists of quasinilpotents.

The converse implications also hold.
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Theorem (V. Shulman-Y. Turovsky 2014 + Sh. 2019)

Let $A$ be a C*-algebra. The following are equivalent:

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Proposition (Sh. 2019)

If the closure of nilpotents in a C*-algebra $A$ contains a normal element, then $A$ is not residually type I.
Nilpotents

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Proposition (Sh. 2019)

If the closure of nilpotents in a C*-algebra $A$ contains a normal element, then $A$ is not residually type I.
Extensions by compact operators

**Step 1:** to prove that $A$ is type I but not CCR $\Rightarrow F(A)$ is not subhomogeneous.

**Sufficient:** to prove that $A \supset K(H) \Rightarrow F(A)$ is not subhomogeneous.
Extensions by compact operators

$q : B(H) \to B(H)/K(H)$
Extensions by compact operators

$q : B(H) \to B(H)/K(H)$

For $A \subset B(H)$, $q(A)'$ is the commutant of $q(A)$ in the Calkin algebra.
q : \( B(H) \to B(H)/K(H) \)

For \( A \subset B(H) \), \( q(A)' \) is the commutant of \( q(A) \) in the Calkin algebra.

**Strategy:** For \( A \supset K(H) \), an element of \( q(A)' \) gives rise to an element of \( F(A) \).
Extensions by compact operators

$q : B(H) \to B(H)/K(H)$

For $A \subset B(H)$, $q(A)'$ is the commutant of $q(A)$ in the Calkin algebra.

**Strategy:** For $A \supset K(H)$, a nilpotent element of $q(A)'$ gives rise to a nilpotent element of $F(A)$.  

Joint work with Dominic Enders

Central sequence algebras via nilpotent elements
Extensions by compact operators

$q : B(H) \to B(H)/K(H)$

For $A \subset B(H)$, $q(A)'$ is the commutant of $q(A)$ in the Calkin algebra.

**Strategy:** Prove that for $A \supset K(H)$, a convergent sequence of nilpotent elements of $q(A)'$ gives rise to a convergent sequence of nilpotent elements of $F(A)$.
q : \( B(H) \to B(H)/K(H) \)

For \( A \subseteq B(H) \), \( q(A)' \) is the commutant of \( q(A) \) in the Calkin algebra.

**Strategy:** Prove that for \( A \supseteq K(H) \), a convergent sequence of nilpotent elements of \( q(A)' \) gives rise to a convergent sequence of nilpotent elements of \( F(A) \).

If a sequence of nilpotent elements of \( q(A)' \) converges to a normal element, then the corresponding sequence of nilpotent elements of \( F(A) \) converges to a normal element.
Extensions by compact operators

\( q : B(H) \to B(H)/K(H) \)

For \( A \subset B(H) \), \( q(A)' \) is the commutant of \( q(A) \) in the Calkin algebra.

**Strategy:** For \( A \supset K(H) \), a convergent sequence of nilpotent elements of \( q(A)' \) gives rise to a convergent sequence of nilpotent elements of \( F(A) \). If a sequence of nilpotent elements of \( q(A)' \) converges to a normal element, then the corresponding sequence of nilpotent elements of \( F(A) \) converges to a normal element.

**Lemma**

Let \( A \subset B(H) \) be a separable C*-algebra, then \( q(A)' \) contains a copy of \( B(H) \).
Step 1: to prove that $A$ is type I but not CCR $\Rightarrow F(A)$ is not subhomogeneous.

Sufficient: to prove that $A \supset K(H) \Rightarrow F(A)$ is not subhomogeneous.

Theorem
Let $A \subset B(H)$ be a separable C*-algebra such that $A \supset K(H)$. Then $F(A)$ is not residually type I.

Joint work with Dominic Enders
Step 1: to prove that $A$ is type I but not CCR $\Rightarrow F(A)$ is not subhomogeneous.

Sufficient: to prove that $A \supset K(H) \Rightarrow F(A)$ is not subhomogeneous.

Theorem
Let $A \subset B(H)$ be a separable C*-algebra such that $A \supseteq K(H)$. Then $F(A)$ is not residually type I. In particular $F(A)$ is not type I and not RFD.
Step 1: to prove that $A$ is type I but not CCR $\Rightarrow F(A)$ is not subhomogeneous.

Sufficient: to prove that $A \supset K(H) \Rightarrow F(A)$ is not subhomogeneous.
**Question:** When is $F(A)$ abelian? $n$-subhomogeneous?

Can assume $A$ is type I.
**Question:** When is $F(A)$ abelian? $n$-subhomogeneous?

Can assume $A$ is type I.

Can assume $A$ is CCR.
Fell’s condition

An element $a \in A$ has global rank not larger than $n$ if for each irreducible representation $\pi$ of $A$, $\text{rank} \pi(a) \leq n$.

A $C^*$-algebra $A$ is a Fell algebra (or satisfies Fell’s condition) if it is generated by elements of global rank 1.

Joint work with Dominic Enders
**Definition** An element $a \in A$ has *global rank not larger than* $n$ if for each irreducible representation $\pi$ of $A$

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**Definition** A C*-algebra $A$ is a *Fell algebra* (or *satisfies Fell’s condition*) if it is generated by elements of global rank 1.
Example 1

\[ A = \{ f \in C([0, 1], M_2) \mid f(1) \in \mathbb{C}1 \} \]
Fell’s condition

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Irreducible representations: 2-dim \( \pi_t = \text{ev}_t, \ t < 1 \), and one 1-dim \( \pi_1 \) corresponding to \( t = 1 \).
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Suppose \( f(1) \neq 0 \). Then \( \text{rank } f(1) = 2 \).
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Irreducible representations: 2-dim \( \pi_t = ev_t, t < 1 \), and one 1-dim \( \pi_1 \) corresponding to \( t = 1 \).

Suppose \( f(1) \neq 0 \). Then \( \text{rank } f(1) = 2 \).

Since rank is lower semicontinuous,

\[ \text{rank } f(t) = 2, \text{ when } t \approx 1. \]
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\[ \text{rank } \pi_t(f) = 2, \text{ when } t \approx 1. \]

Conclusion: \( f \) has global rank 1 \( \Rightarrow f(1) = 0 \).
Hence \( A \) is not a Fell algebra.

Joint work with Dominic Enders

Central sequence algebras via nilpotent elements
Example 2

\[ A = \{ f \in C([0, 1], M_2) \mid f(1) \text{ is diagonal} \} \]
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A is a Fell algebra.
Fell’s condition of higher order

**Definition** A C*-algebra $A$ satisfies *Fell’s condition of order* $n$ if it is generated by elements of global rank $n$. 

Example 3

$A = \{ f \in C([0,1], M_3) | f(1) = \begin{pmatrix} \lambda & \lambda & \mu \\ \end{pmatrix}, \lambda, \mu \in C \}$

$A$ satisfies Fell’s condition of order 2 but not of order 1.
Fell’s condition of higher order

**Definition** A C*-algebra $A$ satisfies *Fell’s condition of order* $n$ if it is generated by elements of global rank $n$.

**Example 3**

$$A = \{ f \in C([0, 1], M_3) \mid f(1) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \lambda, \mu \in \mathbb{C} \}$$
Fell’s condition of higher order

**Definition** A C*-algebra $A$ satisfies *Fell’s condition of order* $n$ if it is generated by elements of global rank $n$.

**Example 3**

$$A = \{ f \in C([0,1], M_3) \mid f(1) = \begin{pmatrix} \lambda & \lambda \\ \lambda & \mu \\ \mu & \end{pmatrix}, \lambda, \mu \in \mathbb{C} \}$$

$A$ satisfies Fell’s condition of order 2 but not of order 1.
Example 4 A (unital) CCR-algebra need not satisfy Fell’s condition of any order.
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Consider the UHF algebra

$$\mathbb{C} \subset M_2 \subset M_4 \subset \ldots \subset M_{2^n},$$

and its telescopic algebra

$$T(M_{2^n}) = \{ f \in C([0, \infty), M_{2^n}) | t \leq i \Rightarrow f(t) \in M_{2^i} \}.$$
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Consider the UHF algebra

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Let

$$A = \{ f \in T(\mathcal{M}_{2^\infty}) \mid f(\infty) \in \mathbb{C}1\}.$$
**Step 2:** For CCR-algebras, 
\[ F(A) \text{ } n\text{-}subhomogeneous \iff A \text{ satisfies Fell’s condition of order } n \]
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Will show strategy for $\Leftarrow$. 
Fell’s condition of order $n \Rightarrow F(A)$ is $n$-subhomogeneous

Observation: If in $B(H)$ we have $x^2 = 0$, $e \geq 0$ is of rank 1, then $[x, e] = 0 \Rightarrow ex = 0, xe = 0$.

Lemma: For any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $e \in (B(H))^+$ with rank $e = 1$ and $x \in (B(H))_1$ with $x^2 = 0$, then $\| [x, e] \| \leq \delta \Rightarrow \| ex \| \leq \epsilon$ and $\| xe \| \leq \epsilon$.

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Central sequence algebras via nilpotent elements
Fell’s condition of order \( n \) \( \Rightarrow \) \( F(A) \) is \( n \)-subhomogeneous

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Fell’s condition of order $n \implies F(A)$ is $n$-subhomogeneous

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**Lemma**

For any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $e \in (B(H))_{+,1}$ with $\text{rank } e \leq N$ and $x \in (B(H))_1$ with $x^{N+1} = 0$, then

$$\|[x, e]\| \leq \delta \implies \|ex^N\| \leq \epsilon \text{ and } \|x^Ne\| \leq \epsilon.$$
Fell’s condition of order $N \Rightarrow F(A)$ is $N$-subhomogeneous

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Central sequence algebras via nilpotent elements
Fell’s condition of order $N \Rightarrow F(A)$ is $N$-subhomogeneous

$A$ is generated by elements of global rank $N$. 

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Fell’s condition of order $N$ $\Rightarrow$ $F(A)$ is $N$-subhomogeneous

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Suppose $x \in F(A)$ with $x^{N+1} = 0$, $\|x\| \leq 1$. 

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Suppose $x \in F(A)$ with $x^{N+1} = 0$, $\|x\| \leq 1$.
Lift it to a central sequence $(x_1, x_2, \ldots)$ with $x_i^{N+1} = 0$, $\|x_i\| \leq 1$. 

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Let $e \geq 0$ be of global rank $N$.

\[
[x_i, e] \to 0
\]

\[
[\rho(x_i), \rho(e)] \to 0
\]

\[
\rho(x_i)^N \rho(e) \to 0, \quad \rho(e)\rho(x_i)^N \to 0.
\]

\[
x_i^N e \to 0, \quad e x_i^N \to 0
\]

\[
x_i^N a \to 0, \quad ax_i^N \to 0, \text{ for any } a \in A \iff x^N = 0 \text{ in } F(A).
\]
Theorem

Let $A$ be a separable CCR C*-algebra. Then $F(A)$ is $n$-subhomogeneous if and only if $A$ satisfies Fell’s condition of order $n$. 

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Central sequence algebras via nilpotent elements
Thank you!