Hecke algebras of quantum groups
based on joint work with Roland Vergnioux and Christian Voigt

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Classical Hecke pairs

Hecke algebras, originally studied in number-theoretic and representation-theory related contexts, have in the last 20 years become an important theme and source of examples for operator algebras.

Definition

Let $\Gamma$ be a discrete group with a subgroup $\Lambda$. We say that $(\Gamma, \Lambda)$ is a Hecke pair if every right coset of $\Lambda$ intersects finitely many left cosets of $\Lambda$ and every left coset of $\Lambda$ intersects finitely many right cosets of $\Lambda$.

Examples of Hecke pairs:

- $\Lambda$ finite or finite index;
- $\Lambda$ normal;
- $(SL(n, \mathbb{Q}), SL(n, \mathbb{Z}))$;
- $(\mathbb{Q} \rtimes \mathbb{Q}^*_+, \mathbb{Z} \rtimes 1)$.
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Hecke (von Neumann) algebras

Given a Hecke pair \((\Gamma, \Lambda)\) the Hecke algebra of \((\Gamma, \Lambda), \mathcal{H}(\Gamma, \Lambda)\), is the space spanned by characteristic functions of double cosets of \(\Lambda\), with the convolution-like product

\[
(f_1 \star f_2)(\gamma) = \sum_{[g] \in \Gamma/\Lambda} f_1(g)f_2(g^{-1}\gamma), \quad \gamma \in \Gamma, f_1, f_2 \in \mathcal{H}(\Gamma, \Lambda).
\]

and the involution

\[
f^*(\gamma) = \overline{f(\gamma^{-1})}, \quad \gamma \in \Gamma.
\]

The formula

\[
(R(f)\xi)([g]) = \sum_{[k] \in \Gamma/\Lambda} \xi([k])f(k^{-1}g), \quad \xi \in \ell^2(\Gamma/\Lambda), [g] \in \Gamma/\Lambda, f \in \mathcal{H}(\Gamma, \Lambda),
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gives an anti-representation of \(\mathcal{H}(\Gamma, \Lambda)\) on \(\ell^2(\Gamma/\Lambda)\).
The von Neumann algebraic closure of $R(\mathcal{H}(\Gamma, \Lambda))$ inside $B(\ell^2(\Gamma/\Lambda))$ is called the Hecke von Neumann algebra of $(\Gamma, \Lambda)$. The vector state associated with $\delta_\Lambda$ is faithful, and in general non-tracial: the algebra might be of type III, and the modular properties of $\omega_{\delta_\Lambda}$ are determined by the ‘coset counting homomorphism’ $\delta : \Gamma \to \mathbb{Q}$,

$$\delta(\gamma) = L(\gamma)/R(\gamma), \quad \gamma \in \Gamma.$$ 

If $\Lambda$ is normal in $\Gamma$ then $\mathcal{H}(\Gamma, \Lambda)$ is simply $\mathbb{C}[\Gamma/\Lambda]$. Boundedness of Hecke operators is not too difficult to see in general, but becomes almost trivial when $\Lambda$ is finite (as then $\ell^2(\Gamma/\Lambda) \subset \ell^2(\Gamma)$).
Hecke (von Neumann) algebras continued

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Schlichting completion

The Hecke condition is also satisfied for a pair \((G, H)\), where \(G\) is a locally compact group and \(H\) a compact-open subgroup of \(G\). Again one can study \(\mathcal{H}(G, H)\) and (anti-)represent it on \(\ell^2(G/H) \subset L^2(G)\).

Let \((\Gamma, \Lambda)\) be a Hecke pair. Consider the group \(\text{Bij}(\Gamma/\Lambda)\) equipped with the topology of pointwise convergence. We have a natural morphism \(\theta : \Gamma \to \text{Bij}(\Gamma/\Lambda)\). Set

\[
G = \overline{\theta(\Gamma)} \subset \text{Bij}(\Gamma/\Lambda),
\]

\[
H = G \cap \text{Stab}_\Lambda.
\]

Proposition

\(H\) is a compact open subgroup in the locally compact (totally disconnected) group \(G\). The pair \((G, H)\) is called the Schlichting completion of \((\Gamma, \Lambda)\); we have \(\mathcal{H}(G, H) \cong \mathcal{H}(\Gamma, \Lambda)\).
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Discrete quantum groups

\( \Gamma \) – discrete quantum group: studied via its Hopf-von Neumann algebra, equipped with the coproduct and left/right Haar weights,

\[
\ell^\infty(\Gamma) := \prod_{\alpha \in I(\Gamma)} M_{n_\alpha}.
\]

\( I(\Gamma) \) – the set of (equivalence classes) of irreducible corepresentations of \( \Gamma \) (irred. representations of the dual compact quantum group \( \widehat{\Gamma} \)), generating the fusion ring of \( \widehat{\Gamma} \):

\[
\alpha \otimes \beta = \bigoplus_{i \in F} \gamma_i.
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We define \( C[\Gamma] := O(\widehat{\Gamma}) \), \( VN(\Gamma) := L^\infty(\widehat{\Gamma}) \subset B(\ell^2(\Gamma)) \).

We can also study algebras \( c_0(\Gamma), c_c(\Gamma), c(\Gamma) \).... To each \( \alpha \in I(\Gamma) \) we associate a central projection \( p_\alpha \in Z(\ell^\infty(\Gamma)) \) and the quantum dimension \( \dim_q(\alpha) > 0 \).
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...their quantum subgroups...

A discrete quantum subgroup $\Lambda \subset \Gamma$ can be defined via the surjective Hopf*-homomorphism

$$\pi : \ell^\infty(\Gamma) \to \ell^\infty(\Lambda),$$

but we will mostly think about the inclusion

$$I(\Lambda) \subset I(\Gamma)$$

or a central group-like projection $p_\Lambda \in \ell^\infty(\Gamma)$

$$\Delta(p_\Lambda)(p_\Lambda \otimes 1) = p_\Lambda \otimes p_\Lambda.$$

Set

$$\ell^\infty(\Gamma/\Lambda) := \{ a \in \ell^\infty(\Gamma) \mid (1 \otimes p_\Lambda)\Delta(a) = a \otimes p_\Lambda \},$$

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...and coset spaces ...

Given $\alpha \in I(\varGamma)$ set

\[ p_{[\alpha]} := \sum_{\beta \sim \alpha} p_\beta. \]

**Proposition**

Each $p_{[\alpha]}$ is a central projection in $\ell^\infty(\varGamma/\Lambda)$; it need not be minimal(!), but is a finite sum of minimal projections in $\ell^\infty(\varGamma/\Lambda)$. The algebra $\ell^\infty(\varGamma/\Lambda)$ is isomorphic to a direct product of matrix algebras.

Set $c_c(\varGamma/\Lambda) := \text{Lin}\{p_{[\alpha]}\ell^\infty(\varGamma/\Lambda) : \alpha \in I(\varGamma)\}$.

Similarly we have a right version of our equivalence relation:

\[ \alpha \sim \beta \iff \alpha \subset \gamma \otimes \beta \text{ for some } \gamma \in I(\Lambda). \]

Write $I(\varGamma)/\Lambda$, $\Lambda \setminus I(\varGamma)$ and $\Lambda \setminus I(\varGamma)/\Lambda$ for the relevant quotient spaces.
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Write $I(\Gamma)/\Lambda$, $\Lambda \setminus I(\Gamma)$ and $\Lambda \setminus I(\Gamma)/\Lambda$ for the relevant quotient spaces.
Hecke condition and commensurator subgroup

Fix $\Lambda \subset \Gamma$, an inclusion of discrete quantum groups. Define $I(\Gamma')$ as the set of these $\alpha \in I(\Gamma)$ that $[\alpha]_{\sim}$ intersects finitely many classes of $\sim$ and $[\alpha]_{\sim}$ intersects finitely many classes of $\sim$.

**Proposition**

The formula above defines a quantum subgroup $\Gamma' \subset \Gamma$, the commensurator of $\Lambda$ in $\Gamma$. We say that $(\Gamma, \Lambda)$ is a Hecke pair if $\Gamma' = \Gamma$.

We have $c_c(\Lambda \backslash \Gamma'/\Lambda) = c_c(\Gamma'/\Lambda) \cap c_c(\Lambda \backslash \Gamma') = c_c(\Gamma/\Lambda) \cap c_c(\Lambda \backslash \Gamma)$.

The action of $\Gamma$ on $\Gamma/\Lambda$ is given by the restriction of the coproduct

$$\Delta : \ell^\infty(\Gamma/\Lambda) \to \ell^\infty(\Gamma) \otimes \ell^\infty(\Gamma/\Lambda);$$

$(\Gamma, \Lambda)$ is a Hecke pair if and only if this action has finite orbits.
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Combinatorial constants...

For $\alpha \in I(\Gamma)$ set

$$\kappa_\alpha = \dim_q(\bar{\alpha} \otimes \alpha)_\Lambda.$$ 

Then

- $(h_R p_\Lambda \otimes p_\alpha) \Delta(p_\alpha) = \kappa_\alpha p_\alpha$;
- $\kappa_\alpha/\left(\dim_q(\alpha)\right)^2$ depends only on the class $[\alpha]$ in $I(\Gamma)/\Lambda$;
- for any $a \in c_c(\bar{\Gamma}/\Lambda)$, and any choices of representatives $\alpha \in [\alpha]$

$$a = \sum_{[\alpha] \in I(\Gamma)/\Lambda} \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes \text{id})\Delta(p_\Lambda)$$
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and quantum invariant measures

**Proposition**

The formula

\[
\mu(a) = \sum_{[\alpha] \in I(\Gamma)/\Lambda} \kappa_{\alpha}^{-1} h_L(a_\alpha)
\]

for \( a \in c_c(\Gamma/\Lambda) \) defines a unique (up to scaling) \( \Gamma \)-invariant measure on \( c_c(\Gamma/\Lambda) \):

\[
(id \otimes \mu)\Delta(a) = \mu(a)1.
\]

We use it to define \( \ell^2(\Gamma/\Lambda) \).
Quantum Hecke algebras

For \( a, b \in c_c(\Lambda \backslash \Gamma' / \Lambda) = c_c(\Gamma / \Lambda) \cap c_c(\Lambda \backslash \Gamma) \) define

\[
a \ast b := \sum_{[\alpha] \in I(\Gamma)/\Lambda} \kappa_\alpha^{-1}(S^{-1}(a_\alpha) h_R \otimes \text{id}) \Delta(b)
\]

and

\[
a^\# := S(a^*) .
\]

**Proposition**

The formulas above define the *-algebra structure on \( c_c(\Gamma / \Lambda) \cap c_c(\Lambda \backslash \Gamma) \), from now on to be denoted \( \mathcal{H}(\Gamma, \Lambda) \) and called the Hecke algebra of the pair \( (\Gamma, \Lambda) \).

Note that the Hecke algebra depends in fact only on \( (\Gamma', \Lambda) \). If \( \Lambda \) is normal in \( \Gamma \), then \( \mathcal{H}(\Gamma, \Lambda) = \mathbb{C}[\Gamma / \Lambda] \).
Quantum Hecke algebras

For $a, b \in c_c(\Lambda \backslash \Gamma' / \Lambda) = c_c(\Gamma / \Lambda) \cap c_c(\Lambda \backslash \Gamma)$ define

$$a \star b := \sum_{[\alpha] \in I(\Gamma)/\Lambda} \kappa^{-1}_\alpha(S^{-1}(a_\alpha) h_R \otimes id) \Delta(b)$$

and

$$a^\#: = S(a^*) .$$

Proposition

The formulas above define the $*$-algebra structure on $c_c(\Gamma / \Lambda) \cap c_c(\Lambda \backslash \Gamma)$, from now on to be denoted $\mathcal{H}(\Gamma, \Lambda)$ and called the Hecke algebra of the pair $(\Gamma, \Lambda)$.

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Quantum Hecke algebras

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**Proposition**

The formulas above define the \(*\)-algebra structure on \( c_c(\Gamma / \Lambda) \cap c_c(\Lambda \setminus \Gamma) \), from now on to be denoted \( \mathcal{H}(\Gamma, \Lambda) \) and called the **Hecke algebra** of the pair \((\Gamma, \Lambda)\).

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Hecke algebra as the algebra of adjointable endomorphisms

The space \( c_c(\Gamma/\Lambda) \) is a right \( \mathbb{C}[\Gamma] \)-module: for \( a \in c_c(\Gamma/\Lambda), x \in \mathbb{C}[\Gamma], \)

\[
a \cdot x = (x \otimes \text{id})\Delta(a),
\]

where we view \( \mathbb{C}[\Gamma] \) as a linear subspace of \( c_c(\Gamma)^* \). A pre-Hilbert structure on \( c_c(\Gamma/\Lambda) \) is given by the functional \( \mu \); we can thus talk about adjointable endomorphisms of \( c_c(\Gamma/\Lambda), \text{End}_F(c_c(\Gamma/\Lambda)). \)

**Theorem**

The map \( \text{ev}_\Lambda : F \mapsto f := F(\rho_\Lambda) \) defines an antimultiplicative isomorphism from \( \text{End}_F(c_c(\Gamma/\Lambda)) \) to \( \mathcal{H}(\Gamma, \Lambda) \), with inverse bijection \( T \) given by \( T(f)(a) = a \star f, f \in \mathcal{H}(\Gamma, \Lambda), a \in c_c(\Gamma/\Lambda). \)

If \( (\Gamma, \Lambda) \) is a Hecke pair, being adjointable is automatic for a map in \( \text{End}_F(c_c(\Gamma/\Lambda)). \)
Hecke algebra as the algebra of adjointable endomorphisms

The space $c_c(\Gamma/\Lambda)$ is a right $\mathbb{C}[\Gamma]$-module: for $a \in c_c(\Gamma/\Lambda), x \in \mathbb{C}[\Gamma],$

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**Theorem**

The map $\text{ev}_A : F \mapsto f := F(p_A)$ defines an antimultiplicative isomorphism from $\text{End}_c(c_c(\Gamma/\Lambda))$ to $\mathcal{H}(\Gamma, \Lambda)$, with inverse bijection $T$ given by $T(f)(a) = a \star f$, $f \in \mathcal{H}(\Gamma, \Lambda), a \in c_c(\Gamma/\Lambda)$.

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**Theorem**

The map \( \text{ev}_\Lambda : F \mapsto f := F(p_\Lambda) \) defines an antimultiplicative isomorphism from \( \text{End}^\prime_c(\Gamma/\Lambda) \) to \( \mathcal{H}(\Gamma, \Lambda) \), with inverse bijection \( T \) given by \( T(f)(a) = a \star f \), \( f \in \mathcal{H}(\Gamma, \Lambda), a \in c_c(\Gamma/\Lambda) \).

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The map $\text{ev}_\Lambda : F \mapsto f := F(p_\Lambda)$ defines an antimultiplicative isomorphism from $\text{End}_\Gamma'(c_c(\Gamma/\Lambda))$ to $\mathcal{H}(\Gamma, \Lambda)$, with inverse bijection $T$ given by $T(f)(a) = a \star f$, $f \in \mathcal{H}(\Gamma, \Lambda)$, $a \in c_c(\Gamma/\Lambda)$.

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Quantum Hecke algebras  
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Definition

We say that $\beta \in I(\Gamma)$ satisfies the property (RT) if there exists $C_\beta > 0$ such that $\kappa_\gamma \leq C_\beta \kappa_\alpha$ for all $\alpha, \gamma \in I(\Gamma)$ such that $\gamma \subset \alpha \otimes \beta$. We say that the inclusion $\Lambda \subset \Gamma$ satisfies the property (RT) if the property (RT) holds for each $\beta \in I(\Gamma')$.

Examples show that Property (RT) does not hold for all $\beta \in I(\Gamma)$.

Theorem

The operators $T(f)$ are bounded with respect to $\| \cdot \|_{\ell^2(\Gamma/\Lambda)}$ for every $f \in H(\Gamma, \Lambda)$ if and only if the inclusion $\Lambda \subset \Gamma$ satisfies the property (RT).

So does (RT) always hold for inclusions?
Action on $\ell^2(\Gamma/\Lambda)$ and the (RT) condition

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Algebraic quantum groups

Discrete quantum groups are special cases of algebraic quantum groups (which in turn form a subclass of locally compact quantum groups).

Definition

An algebraic quantum group $G$ is defined via a multiplier Hopf $\ast$-algebra $O_c(G)$ equipped with positive left and right Haar weights (so for example $\Delta : O_c(G) \to M(O_c(G) \otimes O_c(G))$).

The natural $C^*$- (or von Neumann-) algebraic completions of $O_c(G)$ fit into the framework of locally compact quantum groups of Kustermans-Vaes.

Examples of algebraic quantum groups:

- discrete quantum groups (with $O_c(G) = c_c(\mathbb{G})$);
- compact quantum groups;
- precisely these classical locally compact groups which admit a compact open subgroup.
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Algebraic compact open quantum subgroups

In general for a locally compact quantum group $G$ a compact open quantum subgroup is defined via a group-like projection $P \in Z(C_0(G))$.

**Proposition (Landstad-Van Daele, Kalantar-Kasprzak-AS)**

Let $G$ be an algebraic quantum group. Given a group-like projection $P \in Z(O_c(G))$, so that

$$\Delta(P)(1 \otimes P) = P \otimes P$$

the space $PO_c(G)$ has a natural CQG-algebra structure; we write $PO_c(G) := O(H)$ and call $H$ an algebraic compact open quantum subgroup of $G$.

In that case we can also define the homogeneous space $O_c(G/H)$, which turns out to be an algebraic direct sum of matrix algebras.
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Hecke algebras in compact-open setup

Algebras $\mathcal{O}_c(G)$ admit also a convolution type product $\star$ and involution $\#$, transported via the Fourier transform from the dual multiplier Hopf*-algebra.

Given a compact open quantum subgroup $\mathcal{H}$ associated with $P_\mathcal{H} \in \mathcal{Z}(\mathcal{O}_c(G))$ define:

$$c_c(\mathcal{H}\backslash G/\mathcal{H}) = P_\mathcal{H} \star \mathcal{O}_c(G) \star P_\mathcal{H}.$$ 

**Proposition**

The convolution type product $\star$ and involution $\#$ give the *-algebra structure on $c_c(\mathcal{H}\backslash G/\mathcal{H})$, from now on to be denoted $\mathcal{H}(G, \mathcal{H})$ and called the Hecke algebra of the pair $(G, \mathcal{H})$. We have a natural anti *-isomorphism of $\mathcal{H}(G, \mathcal{H})$ with $\text{End}_G(c_c(G/\mathcal{H}))$; and the elements of the latter algebra act boundedly on $\ell^2(G/\mathcal{H})$.

Let $\mathcal{H}$ be a compact quantum group, and $G$ its Drin'feld double. Then

- $\mathcal{H}$ becomes a compact open quantum subgroup of $G$;
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Quantum Schlichting completion

Let $(\Gamma, \Lambda)$ be a Hecke pair of discrete quantum groups. Let $O_c(G)$ denote the subalgebra of $\ell^\infty(\Gamma)$ generated by the elements $a \star b$ with $a \in c_c(\Gamma/\Lambda)$, $b \in c_c(\Lambda\backslash\Gamma)$.

Theorem

The algebra $O_c(G)$, with the coproduct inherited from $\ell^\infty(\Gamma)$ defines an algebraic quantum group. The projection $p_\Lambda$ belongs to $O_c(G)$ and defines $\mathbb{H}$, a compact open quantum subgroup of $G$. We have compatible isomorphisms $\mathcal{H}(\Gamma, \Lambda) \cong \mathcal{H}(G, \mathbb{H})$, $\ell^2(\Gamma/\Lambda) \cong \ell^2(G/\mathbb{H})$.

Corollary

If $(\Gamma, \Lambda)$ is a Hecke pair, then Hecke operators act boundedly on $\ell^2(\Gamma/\Lambda)$ (so the inclusion $\Lambda \subset \Gamma$ satisfies the (RT) property).

We naturally have a quantum group morphism $\Gamma \to G$ with dense image (i.e. an injective Hopf $\ast$-morphism from $O_c(G)$ to $c(\Gamma)$).
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\[
\mathcal{H}(\Gamma, \Lambda) \cong \mathcal{H}(G, \mathcal{H}), \quad \ell^2(\Gamma/\Lambda) \cong \ell^2(G/\mathcal{H}).
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Reduction procedure

**Definition**
The pair \((\Gamma, \Lambda)\) is **reduced** if the canonical action of \(\Gamma\) on \(\Gamma/\Lambda\) is faithful.

**Proposition**
Given a pair \((\Gamma, \Lambda)\) there exists a canonical reduced pair \((\tilde{\Gamma}, \tilde{\Lambda})\). The pair \((\Gamma, \Lambda)\) satisfies the Hecke condition if and only if \((\tilde{\Gamma}, \tilde{\Lambda})\) does, and if this is the case, the corresponding Hecke algebras are isomorphic.

The notion of being reduced may be used to give an abstract universal property for quantum Schlichting completions.

**Proposition**
Let \((\Gamma, \Lambda)\) be a Hecke pair, \((\tilde{\Gamma}, \tilde{\Lambda})\) its reduction and \((G, H)\) its Schlichting completion. Then \(G\) is discrete if and only if \(\tilde{\Lambda}\) is finite.
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*Given a pair \((\Gamma, \Lambda)\) there exists a canonical reduced pair \((\tilde{\Gamma}, \tilde{\Lambda})\). The pair \((\Gamma, \Lambda)\) satisfies the Hecke condition if and only if \((\tilde{\Gamma}, \tilde{\Lambda})\) does, and if this is the case, the corresponding Hecke algebras are isomorphic.*

The notion of being reduced may be used to give an abstract universal property for quantum Schlichting completions.

Proposition
Let \((\Gamma, \Lambda)\) be a Hecke pair, \((\tilde{\Gamma}, \tilde{\Lambda})\) its reduction and \((G, H)\) its Schlichting completion. Then \(G\) is discrete if and only if \(\tilde{\Lambda}\) is finite.
Reduction procedure

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Hecke von Neumann algebras revisited

Let us restate the consequences of the last results:

**Theorem**

Given a Hecke pair of discrete quantum groups $(\Gamma, \Lambda)$ we have a natural anti-representation $R : \mathcal{H}(\Gamma, \Lambda) \to B(\ell^2(\Gamma/\Lambda))$. Thus we can define the Hecke von Neumann algebra of $(\Gamma, \Lambda)$ setting

$$H(\Gamma, \Lambda) := R(\mathcal{H}(\Gamma, \Lambda))''.$$ 

The von Neumann algebra $H(\Gamma, \Lambda)$ admits a natural faithful vector state given by the vector $\delta_\Lambda$. The modular theory of the pair $(H(\Gamma, \Lambda), \omega_{\delta_\Lambda})$ can be described explicitly in terms of the modular element $\nabla \in c(\Lambda \setminus \Gamma/\Lambda)$.

- we have an explicit formula for $\nabla_{[[\alpha]]}$ for any $[[\alpha]] \in \Lambda \setminus I(\Gamma)/\Lambda$, in general involving the $\kappa$ constants, the modular theory of $\Gamma$ and the ‘counting coset quotients’;

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Quantum HNN extensions

**Definition (Fima)**

Let $\Gamma_0$ be a discrete quantum group with two quantum subgroups $\Lambda_\epsilon \subset \Gamma_0$ ($\epsilon = \pm 1$) and a Hopf $\ast$-algebra isomorphism $\theta : \mathbb{C}[\Lambda_1] \to \mathbb{C}[\Lambda_{-1}]$. Define the discrete quantum group $\Gamma = HNN(\Gamma_0, \theta)$ by setting $\mathbb{C}[[\Gamma]]$ to be generated by $\mathbb{C}[[\Gamma_0]]$ and a group-like unitary $w$ such that $w^\epsilon b w^{-\epsilon} = \theta^\epsilon(b)$ for $b \in \mathbb{C}[[\Lambda_\epsilon]]$.

**Proposition**

Assume that $\Lambda_\epsilon$ have finite index in $\Gamma_0$ and at least one of them is distinct from $\Gamma_0$. Then $\Gamma_0$ is commensurated in $\Gamma$, not normal, and of infinite index.

The map $\theta$ induces a partial map on $I(\Gamma)$, denoted by the same symbol.

**Proposition**

Assume that $\bigcap_{k \in \mathbb{Z}} \text{Dom } \theta^k = \{1\}$. Then $\Gamma \curvearrowright \Gamma/\Gamma_0$ is faithful.
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Quantum HNN extensions – concrete class of examples

Let $\Sigma_{\pm 1}$ be finite quantum groups (for instance duals of classical finite groups). Set

$$\Gamma_0 = \prod_{k \in \mathbb{Z}^*} \Sigma_{\text{sgn}(k)}$$

and consider the finite index subgroups

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- the modular function $\nabla$ of the Hecke pair $(\Gamma, \Gamma_0)$ is non trivial as soon as $\Sigma_1, \Sigma_{-1}$ have different cardinalities;
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Questions to look at

We have developed a general framework to study Hecke (operator) algebras associated with quantum groups; but a lot of work remains to be done! Here is just a sample of questions.

- what is a purely combinatorial/categorical proof of Property (RT) for Hecke inclusions?

- suppose that $G$ is an algebraic quantum group and $H$ its compact open subgroup (given by a group-like projection $P_H \in Z(C_0(G))$). Is $H$ necessarily an algebraic quantum subgroup, i.e. do we have $P_H \in O_c(G)$?

- in what sense is the quantum Schlichting completion $G$ totally disconnected?

- How do we construct, or better, ‘observe in nature’ more examples? In general we do not have a good/rich supply of discrete quantum subgroups...
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References

**Classical Hecke algebras and Schlichting completions:**

K.Tzanev, Hecke $C^*$-algebras and amenability, JOT, 2003

**This talk:**
Invitation

Conference **Noncommutative harmonic analysis and quantum groups**

**Będlewo (Poland), 11th September – 16th July 2022**


**Organisers**: Kenny De Commer, Jacek Krajczok, Adam Skalski

**Confirmed speakers**: Michael Brannan, Matthew Daws, Amaury Freslon, Mehrdad Kalantar, Paweł Kasprzak, Anna Kula, Stefaan Vaes, Roland Vergnioux, Christian Voigt,...