QUANTUM SU(2)

LET $q \in]0,1]$. COORDINATE ALG. $\mathcal{O}(SU_q(2))$

UNIVERSAL UNITAL $*$-ALG GEN a & b S.T.

& $u = \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix}$ UNITARY FUNDAMENTAL.

HOPF $*$-ALG: COPRODUCT $\Delta(u) = u \otimes u$, ANTIPODE

$S(u) = u^*$, & COUNIT $\epsilon(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

THM HAAR STATE: $h : \mathcal{O}(SU_q(2)) \longrightarrow \mathbb{C}$

UNIQUE LIN. MAP S.T. $h(1) = 1$ & INVARIANT

$$(h \otimes 1)\Delta(x) = h(x) \cdot 1 = (1 \otimes h)\Delta(x).$$

INNER PRODUCT $\langle x, y \rangle := h(x^* \cdot y)$ HILBERT SPACE

COMPLETION $L^2(SU_q(2))$. INJECTIVE UNITAL $*$-HOM

$\rho : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{B}(L^2(SU_q(2)))$ "LEFT MULTIPLICATION".

C^*-NORM $\|x\| := \|\rho(x)\|_\infty$ C^*-COMPLETION

$C(SU_q(2))$ C^*-ALG. QUANTUM SU(2)

COMPACT C^*-ALG QUANTUM GRP.

TWISTED DERIVATIONS & SEMINORMS

LET $t \in]0,1]$. ALG. AUTOMORPHISM $\sigma_L^t : \mathcal{O}(SU_q(2)) \circlearrowleft$

$$\sigma_L^t(u) = u \cdot \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix} \qquad \sigma_L^t(x^*) = \left((\sigma_L^t)^{-1}(x)\right)^*.$$

TWISTED DERIVATIONS $\partial_1, \partial_2, \partial_3^t : \mathcal{O}(SU_q(2)) \circlearrowleft$

$$\partial_1(u) = u \cdot \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix} \qquad \partial_i(x \cdot y) = \partial_i(x) \cdot \sigma_L^q(y) + (\sigma_L^q)^{-1}(x) \cdot \partial_i(y)$$

$$\partial_2(u) = u \cdot \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix} \qquad i = 1, 2$$

$$\partial_3^t(x \cdot y) = \partial_3^t(x) \cdot \sigma_L^t(y) + (\sigma_L^t)^{-1}(x) \cdot \partial_3^t(y)$$

$$\partial_3^t(u) = u \cdot \begin{pmatrix} [-1/2]_t & 0 \\ 0 & [1/2]_t \end{pmatrix}$$

$$[r]_t := \begin{cases} \dfrac{t^r - t^{-r}}{t - t^{-1}} & t \neq 1 \\ r & t = 1 \end{cases} \qquad \forall r \in \mathbb{R}.$$

DFN LEFT DIRAC OPERATION

$$\partial^t = \begin{pmatrix} \partial_3^t & -\partial_1 \\ -\partial_2 & -\partial_3^t \end{pmatrix} : \mathcal{O}(SU_q(2)) \longrightarrow M_2(\mathcal{O}(SU_q(2)))$$

WARNING: FOR $t \neq q$ LEFT DIRAC OPERATION NOT A TWISTED DERIVATION.

DFN SEMINORM $L_{t,q} : \mathcal{O}(SU_q(2)) \longrightarrow [0, \infty[$

$$L_{t,q}(x) := \| \partial^t(x) \|_\infty \qquad \text{OP. NORM.}$$

WHY IS ∂^t SO SPECIAL?

1) ∂^t IS q-GEOMETRIC : COMES FROM PAIRING BETWEEN $\mathcal{U}_q(su(2))$ & $\mathcal{O}(SU_q(2))$

2) ∂^t IS SPECTRAL GEOMETRIC : COMES FROM TWISTED COMMUTATORS W. VERTICAL & HORIZONTAL DIRAC OP.

NO GLOBAL DIRAC OPERATOR

$\longrightarrow$ ENLARGE DOMAIN OF $L_{t,q}$ & INTRODUCE $L^{MAX}_{t,q} : LIP_t(SU_q(2)) \longrightarrow [0,\infty[$.

FOR $t = 1$: ∂^1 RELATES TO WORK OF KRÄHMER - RENNIE - SENIOR : NON-TRIVIAL TWISTED HOCHSCHILD COCYCLE IN DEGREE 3.

FOR $t = q$: ALG. AUTOMORPHISM $\sigma_R^q : \mathcal{O}(SU_q(2)) \circlearrowleft$

$$\sigma_R^q(u) := \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \cdot u .$$

TWISTED DERIVATIONS $\delta_1, \delta_2, \delta_3^q : \mathcal{O}(SU_q(2)) \circlearrowleft$

$$\delta_1(u) = \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix} \cdot u$$

$$\delta_i(x \cdot y) = \delta_i(x) \cdot \sigma_R^q(y) + (\sigma_R^q)^{-1}(x)\, \delta_i(y)$$

$$\delta_2(u) = \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix} \cdot u$$

$$\delta_3^q(u) = \begin{pmatrix} [-1/2]_q & 0 \\ 0 & [1/2]_q \end{pmatrix} \cdot u$$

DFN RIGHT DIRAC OPERATION

$$\eth^q = \begin{pmatrix} \eth_3^q & -\eth_1 \\ -\eth_2 & -\eth_3^q \end{pmatrix} : \mathcal{O}(SU_q(2)) \longrightarrow M_2(\mathcal{O}(SU_q(2)))$$

THM $u \cdot \partial^q(x) \cdot u^* = \eth^q(x)$

COR $L_{q,q}(x) = \| \partial^q(x) \|_\infty = \| \eth^q(x) \|_\infty$

COMPACT QUANTUM METRIC SPACES

COMPLETE OPERATOR SYSTEM $X \subseteq \mathbb{B}(H)$

S.T. $1 \in X$ & CLOSED SUBSPACE

$x^* \in X \quad \forall x \in X$.

STATE SPACE $\mathcal{S}(X)$ CONSISTS OF POSITIVE LINEAR MAPS $\mu : X \longrightarrow \mathbb{C}$ S.T. $\mu(1) = 1$.

COMPACT HAUSDORFF SPACE W. WEAK $*$-TOPOLOGY

FUNDAMENTAL OBS: SEMINORM $L : \mathcal{X} \longrightarrow [0, \infty[$

$\mathcal{X} \subseteq X$ NORM-DENSE SUBSPACE

1) $1 \in \mathcal{X}$ & $L(1) = 0$

2) $x^* \in \mathcal{X} \quad \forall x \in \mathcal{X}$ & $L(x^*) = L(x)$.

MONGE-KANTOROVICH METRIC $d_L : \mathcal{S}(X) \times \mathcal{S}(X) \longrightarrow [0, \infty]$

$$d_L(\mu, \nu) := \sup \{ |\mu(x) - \nu(x)| \mid x \in \mathcal{X}, \ L(x) \leq 1 \}$$

DFN [RIEFFEL] (X, L) COMPACT QUANTUM METRIC SPACE WHEN d_L METRIZES WEAK $*$-TOPOLOGY

THM [K. & KYED] $(C(SU_q(2)), L_{t,q})$ & $(C(SU_q(2)), L^{MAX}_{t,q})$ ARE COMPACT QUANTUM METRIC SPACES. FOR $t = q = 1$: $SU(2) \cong S^3$

EMBEDDING ISOMETRY

$$S^3 \hookrightarrow \mathcal{S}(C(SU(2)))$$

$$x \longmapsto EV_x .$$

2 X ROUND METRIC

$$d_{L_{1,1}} = d_{L^{MAX}_{1,1}}$$

QUANTUM GROMOV-HAUSDORFF DISTANCE

(X, L) & (Y, K) CQMS.

DFN Mnn SEMINORM $M : \mathcal{X} \oplus \mathcal{Y} \longrightarrow [0, \infty[$ & ADMISSIBLE $(X \oplus Y, M)$ IS CQMS

$$\begin{tikzcd} & \mathcal{X}_{SA} \oplus \mathcal{Y}_{SA} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{X}_{SA} & & \mathcal{Y}_{SA} \end{tikzcd}$$

$L : \mathcal{X}_{SA} \to [0, \infty[$ & $K : \mathcal{Y}_{SA} \longrightarrow [0, \infty[$ QUOTIENT SEMINORMS.

ISOMETRIC EMBEDDINGS

$$\begin{tikzcd} & \mathcal{S}(X \oplus Y) & \\ \mathcal{S}(X) \arrow[ur, "\pi_1^*"] & & \mathcal{S}(Y) \arrow[ul, "\pi_2^*"'] \end{tikzcd}$$

HAUSDORFF DISTANCE $\quad \mathrm{DIST}_H^{d_M}\left(\mathcal{S}(X), \mathcal{S}(Y)\right)$

<u>DFN</u> [RIEFFEL] QUANTUM GROMOV-HAUSDORFF DISTANCE

$$\mathrm{DIST}_Q\left((X, L) ; (Y, K)\right) := \mathrm{INF}\left\{ \mathrm{DIST}_H^{d_M}\left(\mathcal{S}(X), \mathcal{S}(Y)\right) \;\middle|\; M : \mathcal{X} \oplus \mathcal{Y} \to [0, \infty[\text{ ADMISSIBLE} \right\}$$

<u>THM</u> [RIEFFEL] DIST_Q COMPLETE METRIC ON "ISOMETRIC ISOMORPHISM" CLASSES OF CQMS.

<u>THM</u> [K. & KYED] $\mathrm{DIST}_Q\left(\left(C(SU_q(2)), L_{t,q}\right) ; \left(C(SU_q(2)), L_{t,q}^{MAX}\right)\right) = 0 \quad \forall t, q \in]0,1]$

<u>THM</u> [K. & KYED] $]0,1] \times]0,1] \longrightarrow \left(\mathrm{CQMS}/_\sim, \mathrm{DIST}_Q \right)$

$(t, q) \longmapsto \left(C(SU_q(2)), L_{t,q}^{MAX} \right)$ CONTINUOUS.

<u>CHALLENGE</u>: PROVE SIMILAR RESULT IN LATRÉMOLIÈRE'S CONTEXT (E.G. QUANTUM GROMOV-HAUSDORFF PROPINQUITY)