

QUANTUM $SU(2)$

LET $q \in]0, 1[$.

COORDINATE ALG. $\mathcal{O}(SU_q(2))$

UNIVERSAL UNITAL \ast -ALG GEN a & b S.T.

$$u = \begin{pmatrix} a^* & -q b \\ b^* & a \end{pmatrix} \quad \text{UNITARY FUNDAMENTAL.}$$

HOPF \ast -ALG : COPRODUCT $\Delta(u) = u \otimes u$, ANTIPODE

$$S(u) = u^*, \quad \text{COUNIT } \epsilon(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

THM HAAR STATE : $h : \mathcal{O}(SU_q(2)) \longrightarrow \mathbb{C}$

UNIQUE LIN. MAP S.T. $h(1) = 1$ & INVARIANT

$$(h \otimes 1) \Delta(x) = h(x) \cdot 1 = (1 \otimes h) \Delta(x).$$

INNER PRODUCT $\langle x, y \rangle := h(x^* \cdot y)$ HILBERT SPACE

COMPLETION $L^2(SU_q(2))$ INJECTIVE UNITAL \ast -HOM

$\mathcal{D} : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{B}(L^2(SU_q(2)))$ "LEFT MULTIPLICATION"

C^* -NORM $\|x\| := \|\mathcal{D}(x)\|_\infty$ C^* -COMPLETION

$C(SU_q(2))$ C^* -ALG. QUANTUM $SU(2)$

COMPACT C^* -ALG QUANTUM GRP.

TWISTED DERIVATIONS & SEMINORMS

LET $t \in]0, 1[$. ALG. AUTOMORPHISM $\sigma_L^t : \mathcal{O}(SU_q(2)) \curvearrowright$

$$\sigma_L^t(u) = u \cdot \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix}$$

$$\sigma_L^t(x \cdot *) = \left((\sigma_L^t)^{-1}(x) \right) *$$

TWISTED DERIVATIONS

$\partial_1, \partial_2, \partial_3^t : \mathcal{O}(SU_q(2)) \curvearrowright$

$$\partial_1(u) = u \cdot \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}$$

$$\partial_i(x \cdot y) = \partial_i(x) \cdot \sigma_L^q(y)$$

$$+ (\sigma_L^q)^{-1}(x) \cdot \partial_i(y)$$

$$\partial_2(u) = u \cdot \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix}$$

$i = 1, 2$

$$\partial_3^t(x \cdot y) = \partial_3^t(x) \cdot \sigma_L^t(y)$$

$$+ (\sigma_L^t)^{-1}(x) \cdot \partial_3^t(y)$$

$$\partial_3^t(u) = u \cdot \begin{pmatrix} [-1/2]_t & 0 \\ 0 & [1/2]_t \end{pmatrix}$$

$$[r]_t := \begin{cases} \frac{t^r - t^{-r}}{t - t^{-1}} & t \neq 1 \\ r & t = 1 \end{cases}$$

$\forall r \in \mathbb{R}$.

DEF LEFT DIRAC OPERATION

$$\partial^t = \begin{pmatrix} \partial_3^t & -\partial_1 \\ -\partial_2 & -\partial_3^t \end{pmatrix}$$

$$: \mathcal{O}(SU_q(2)) \longrightarrow M_2(\mathcal{O}(SU_q(2)))$$

WARNING: FOR $t \neq q$ @ LEFT DIRAC OPERATION

NOT A TWISTED DERIVATION.

DEF SEMINORM $L_{t,q} : \mathcal{O}(SU_q(2)) \longrightarrow [0, \infty[$

$$L_{t,q}(x) := \| \partial^t(x) \|_\infty \quad \text{OP. NORM.}$$

WHY IS ∂^t SO SPECIAL?

1) ∂^t IS q -GEOMETRIC : COMES FROM PAIRING

BETWEEN $U_q(\mathfrak{su}(2))$ & $\mathcal{O}(SU_q(2))$

2) ∂^t IS SPECTRAL GEOMETRIC : COMES FROM TWISTED

COMMUTATORS W. VERTICAL & HORIZONTAL DIRAC OP.

NO GLOBAL DIRAC OPERATOR

→ ENLARGE DOMAIN OF $L_{t,q}$ &

INTRODUCE $L_{t,q}^{\text{MAX}} : \text{LIP}_t(SU_q(2)) \rightarrow [0, \infty[$

FOR $t=1$: ∂^1 RELATES TO WORK OF

KRÄHMER - RENNIE - SENIOR : NON-TRIVIAL TWISTED HOCHSCHILD

COCYCLE IN DEGREE 3.

FOR $t=q$: ALG. AUTOMORPHISM $\sigma_R^q : \mathcal{O}(SU_q(2)) \curvearrowright$

$\sigma_R^q(u) := \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \cdot u$ TWISTED DERIVATIONS $\delta_1, \delta_2, \delta_3$

$\delta_1(u) = \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix} \cdot u$

$\delta_2(u) = \begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix} \cdot u$

$\delta_3(u) = \begin{pmatrix} [-1/2]_q & 0 \\ 0 & [1/2]_q \end{pmatrix} \cdot u$

: $\mathcal{O}(SU_q(2)) \curvearrowright$

$\delta_i(x \cdot y) = \delta_i(x) \cdot \sigma_R^q(y) + (\sigma_R^q)^{-1}(x) \cdot \delta_i(y)$

DEF RIGHT DIRAC OPERATION

$$\delta^q = \begin{pmatrix} \delta_3^q & -\delta_1 \\ -\delta_2 & -\delta_3^q \end{pmatrix} : \mathcal{O}(SU_7(\mathbb{Z})) \longrightarrow M_2(\mathcal{O}(SU_7(\mathbb{Z})))$$

THM $u \cdot \partial^q(x) \cdot u^* = \delta^q(x)$

COR $L_{q,q}(x) = \|\partial^q(x)\|_\infty = \|\delta^q(x)\|_\infty$

COMPACT QUANTUM METRIC SPACES

COMPLETE OPERATOR SYSTEM

$$X \subseteq \mathcal{B}(H)$$

s.t. $1 \in X$ &

CLOSED SUBSPACE

$$x^* \in X \quad \forall x \in X.$$

STATE SPACE $\mathcal{P}(X)$ CONSISTS OF POSITIVE LINEAR

MAPS $\mu : X \longrightarrow \mathbb{C}$

s.t. $\mu(1) = 1.$

COMPACT HAUSSDORFF SPACE

w. WEAK *-TOPOLOGY

FUNDAMENTAL OBS: SEMINORM

$$L : X \longrightarrow [0, \infty[$$

1) $1 \in X$ & $L(1) = 0$

\cap
 X NORM-DENSE SUBSPACE

2) $x^* \in X \quad \forall x \in X$ & $L(x^*) = L(x).$

MONGE-KANTOROVICH METRIC

$$d_L : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0, \infty]$$

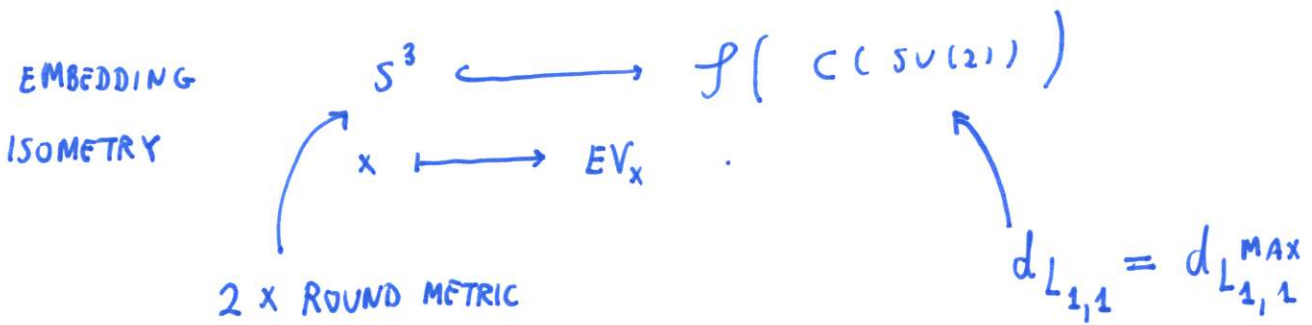
$$d_L(\mu, \nu) := \sup \left\{ |\mu(x) - \nu(x)| \mid x \in X, L(x) \leq 1 \right\}$$

DEF [RIEFFEL] (X, L) COMPACT QUANTUM METRIC SPACE WHEN d_L METRIZES WEAK *-TOPOLOGY

THM [K. & KYED] $(C(SU_q(2)), L_{t,q})$ &

$(C(SU_q(2)), L_{t,q}^{MAX})$ ARE COMPACT QUANTUM

METRIC SPACES. FOR $t = q = 1$: $SU(2) \cong S^3$

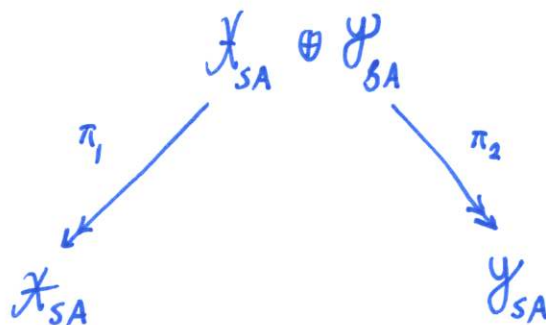


QUANTUM GROMOV-HAUSDORFF DISTANCE

(X, L) & (Y, K) CQMS.

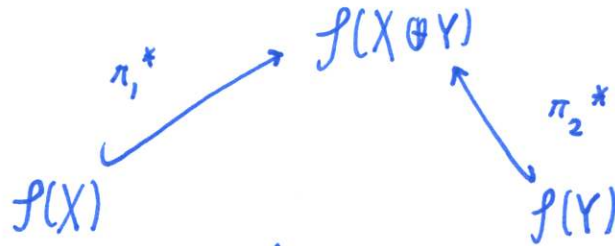
DEF M SEMINORM $M: X \oplus Y \rightarrow [0, \infty]$ &

ADMISSIBLE $(X \oplus Y, M)$ IS CQMS



$L: X_{SA} \rightarrow [0, \infty]$ & $K: Y_{SA} \rightarrow [0, \infty]$ QUOTIENT SEMINORMS.

ISOMETRIC EMBEDDINGS



HAUSDORFF DISTANCE

$$d_M \text{ DIST}_H(f(X), f(Y))$$

DEF [RIEFFEL] QUANTUM GROMOV-HAUSDORFF DISTANCE

$$\text{DIST}_Q((X, L); (Y, K)) := \inf \left\{ \text{DIST}_H^{d_M}(f(X), f(Y)) \mid M: X \oplus Y \rightarrow [0, \infty] \text{ ADMISSIBLE} \right\}$$

THM [RIEFFEL] DIST_Q COMPLETE METRIC ON

"ISOMETRIC ISOMORPHISM" CLASSES OF CQMS.

$$\text{THM [K. & KYED]} \quad \text{DIST}_Q \left((C(SU_q(2)), L_{t,q}); (C(\otimes SU_q(2)), L_{t,q}^{\text{MAX}}) \right) = 0 \quad \forall t, q \in]0, 1[$$

$$\text{THM [K. & KYED]} \quad]0, 1[\times]0, 1[\longrightarrow \left(\text{CQMS} / \sim, \text{DIST}_Q \right)$$

$$(t, q) \longmapsto \left(C(SU_q(2)), L_{t,q}^{\text{MAX}} \right) \quad \text{CONTINUOUS.}$$

CHALLENGE: PROVE SIMILAR RESULT IN LATRÉMOLIÈRE'S CONTEXT (E.G. QUANTUM GROMOV-HAUSDORFF PROPINQUITY)