The gauge group and perturbation semigroup of an operator system

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Background information about operator systems

Definition

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$. A concrete operator system is a (usually closed) linear subspace $\mathcal{E}$ of $B(\mathcal{H})$ that is closed under the involution, i.e., $x \in \mathcal{E}$ implies $x^* \in \mathcal{E}$.

Background information about operator systems

Definition

Let $\mathcal{E}$ be an operator system, $\varphi : \mathcal{E} \to \mathcal{E}$ be a linear map, and φ_n be the induced map $\varphi_n : M_n(\mathcal{E}) \to M_n(\mathcal{E})$.

1. The map φ is called completely bounded if $\sup\limits_{n>0} ||\varphi_n|| < \infty$, and we set
$$||\varphi||_{cb} = \sup_{n>0} ||\varphi_n||.$$
2. The map φ is called $n-$positive if φ_n is positive, and φ is called completely positive if φ_n is $n-$positive for all $n > 0$.

In addition if φ is unital we call it a unital completely positive(UCP) map.

Background information about operator systems

Fact about a UCP map φ

Let $\{V_i\}_{i\leq k} \subset B(\mathcal{H})$ such that $\sum V_i^* V_i = \mathrm{Id}$. Then the map

$$\begin{aligned} \varphi : B(\mathcal{H}) &\to B(\mathcal{H}) \\ x &\mapsto \sum V_i^* x V_i \end{aligned}$$

is a unital completely positive map.

The gauge group of an operator system

We can embed $\mathcal{E}$ into some C^*-algebra $\mathcal{A}$, and then take the gauge group of $\mathcal{E}$ as the collection of unitary elements of $\mathcal{A}$ that keep $\mathcal{E}$ invariant under the unitary transformation, i.e.,

$$\mathcal{G}(\mathcal{E}) := \{u \in \mathcal{A} : u^* \mathcal{E} u \subset \mathcal{E}\}.$$

The C^*-algebra $\mathcal{A}$ can be taken as

1. the C^*-envelope,
2. the injective envelope,
3. the C^*-algebra $C^*(\mathcal{E})$ generated by $\mathcal{E}$.
4. else

The gauge group of an operator system

Definition

We define the gauge group $\mathcal{G}(\mathcal{E})$ of $\mathcal{E}$ as

$$\mathcal{G}(\mathcal{E}) := \{U \in \mathcal{U}(C^*(\mathcal{E})) \,\Big|\, U^*\mathcal{E}U \subset \mathcal{E}\},$$

here $\mathcal{U}(C^*(\mathcal{E}))$ denotes the group of all the unitary elements in $C^*(\mathcal{E})$.

The gauge group of an operator system

Definition

We denote by $\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$ the collection of rank-1 unital completely positive maps, i.e.,

$$\mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E}) := \left\{ \varphi : \mathcal{E} \to \mathcal{E} \,\middle|\, \varphi(\cdot) = V^*(\cdot)V \text{ for some } V \in B(\mathcal{H}) \text{ with } V^*V = \mathrm{Id} \right\}.$$

Proposition

There is a multiplicative map $\Psi : \mathcal{G}(\mathcal{E}) \to \mathrm{UCP}_{\mathrm{rank}=1}(\mathcal{E})$ *defined as*

$$\Psi : U \mapsto U^*(\cdot)U, \quad U \in \mathcal{G}(\mathcal{E}).$$

Perturbation Semigroup of an operator system

Inspired by the definition of perturbation semigroups introduced in [CCvS13, NvS16, Hes16], we define the perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ of an operator system as follows:

Definition

Let $\mathcal{E}$ be an operator system, we define the perturbation semigroup $\mathrm{Pert}(\mathcal{E})$ as the collection of all the finite sums of the form $\sum a_i \otimes b_i^\circ \in C^*(\mathcal{E}) \otimes C^*(\mathcal{E})^\circ$ satisfying the following requirements:

1. $\sum a_i b_i = \mathrm{Id}$,
2. $\sum a_i \mathcal{E} b_i \subset \mathcal{E}$,
3. $\sum a_i \otimes b_i^\circ = \sum b_i^* \otimes a_i^{*\circ}$.

Here $C^*(\mathcal{E})^\circ$ denotes the opposite algebra of $C^*(\mathcal{E})$ and $b_i^\circ, a_i^{*\circ} \in C^*(\mathcal{E})^\circ$.

Perturbation Semigroup of an operator system

We denote by $\mathrm{UCBH}(\mathcal{E})$ the collection of all unital completely bounded Hermitian maps over $\mathcal{E}$, i.e.,

$$\mathrm{UCBH}(\mathcal{E}) := \left\{\Psi : \mathcal{E} \to \mathcal{E} \middle| \Psi(x^*) = \Psi(x)^*, \Psi(\mathrm{Id}) = \mathrm{Id}, \Psi \text{ is completely bounded}\right\}.$$

Proposition ([Don21])

There is a semigroup homomorphism Φ *from* $\mathrm{Pert}(\mathcal{E})$ *to* $\mathrm{UCBH}(\mathcal{E})$ *defined by*

$$\Phi : \mathrm{Pert}(\mathcal{E}) \to \mathrm{UCBH}(\mathcal{E})$$
$$\omega \mapsto \sum a_i(\cdot)b_i$$

with $\omega = \sum a_i \otimes b_i^\circ \in \mathrm{Pert}(\mathcal{E})$.

Perturbation Semigroup of an operator system

We denote by $\overline{\mathrm{Pert}(\mathcal{E})}$ the closure of $\mathrm{Pert}(\mathcal{E})$ with respect to the Haagerup tensor norm $||\cdot||_h$.

Proposition ([Don21])

Let $\mathcal{E} \subset B(\mathcal{H})$ be a unital operator system, the map $\Phi : \mathrm{Pert}(\mathcal{E}) \to \mathrm{UCBH}(\mathcal{E})$ can be extended to a map

$$\widetilde{\Phi} : \overline{\mathrm{Pert}(\mathcal{E})} \to \mathrm{UCBH}(\mathcal{E}),$$

such that $\widetilde{\Phi}\big|_{\mathrm{Pert}(\mathcal{E})} = \Phi$. Moreover, if we equip $\overline{\mathrm{Pert}(\mathcal{E})}$ and $\mathrm{UCBH}(\mathcal{E})$ with the metric topology induced by Haagerup tensor norm $||\cdot||_h$ and complete bound norm $||\cdot||_{cb}$ respectively, the map $\widetilde{\Phi}$ is contractive.

Perturbation Semigroup of an operator system

Example

Let $\{E_{ij}\}$, $1 \leq i, j \leq 2$ be the standard matrix units for $M_2(\mathbb{C})$. Define

$$\mathrm{Toep}_2 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \subset M_2(\mathbb{C}) \right\}.$$

Take $\omega_1, \omega_2 \in \mathrm{Pert}(\mathrm{Toep}_2)$ given as

$$\omega_1 = E_{12} \otimes E_{12}^\circ + E_{21} \otimes E_{21}^\circ + E_{11} \otimes E_{11}^\circ + E_{22} \otimes E_{22}^\circ,$$
$$\omega_2 = (E_{12} + E_{21}) \otimes (E_{12} + E_{21})^\circ.$$

By a direct computation we obtain that $\Phi(\omega_1) = \Phi(\omega_2)$ on Toep_2, both give rise to the transposition map on Toep_2.
$||\Phi(\omega_1)||_{cb} = 1 < ||\omega_1||_h = 2$.

Perturbation Semigroup of an operator system

Definition

We denote by $\mathrm{Pert}^+(\mathcal{E})$ the subsemigroup of $\mathrm{Pert}(\mathcal{E})$:

$$\mathrm{Pert}^+(\mathcal{E}) := \{\omega \in \mathrm{Pert}(\mathcal{E}) \,\big|\, \omega = \sum a_i \otimes a_i^{*\circ} \text{ for some } a_i \in C^*(\mathcal{E})\}.$$

Proposition ([Don21])

Let $\overline{\mathrm{Pert}^+(\mathcal{E})}$ *be the closure of* $\mathrm{Pert}^+(\mathcal{E})$ *with respect to Haagerup tensor norm. We can extend the map* $\Phi : \mathrm{Pert}^+(\mathcal{E}) \to \mathrm{UCP}(\mathcal{E})$ *to a map*

$$\widetilde{\Phi} : \overline{\mathrm{Pert}^+(\mathcal{E})} \to \mathrm{UCP}(\mathcal{E}),$$

such that $\widetilde{\Phi}\big|_{\mathrm{Pert}^+(\mathcal{E})} = \Phi$. *Moreover, we have* $||\omega||_h = 1$ *and* $||\widetilde{\Phi}(\omega)||_{cb} = 1$ *for every* $\omega \in \overline{\mathrm{Pert}^+(\mathcal{E})}$.

Gauge group of the Toeplitz system

We denote by Toep_n the Toeplitz system that contains all the $n \times n$ complex Toeplitz matrices T of the form

$$T := \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

with $t_k \in \mathbb{C}$ for $k = -n+1, ..., n-1$.

Gauge group of the Toeplitz system

Proposition ([Don21])

The gauge group $\mathcal{G}(\mathrm{Toep}_n)$ is generated by the diagonal matrices $U_{\alpha,\beta}$ and anti-diagonal matrix V of the form

$$U_{\alpha,\beta} = \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\alpha}^{n-2}\beta^{n-1} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix},$$

here $|\alpha| = |\beta| = 1.$

Gauge group of the Toeplitz system

Corollary ([Don21])

The group of $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$ *is isomorphic to the semidirect product of* $U(1)$ *and* $\mathbb{Z}_2$, *and the gauge group* $\mathcal{G}(\mathrm{Toep}_n)$ *is different from* $\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n)$ *by a phase factor, that is,*

$$\mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n) = U(1) \rtimes \mathbb{Z}_2$$

and

$$\mathcal{G}(\mathrm{Toep}_n) = U(1) \times (U(1) \rtimes \mathbb{Z}_2)\,.$$

Moreover, We have the short exact sequence which is independent of n:

$$1 \longrightarrow U(1) \longrightarrow \mathcal{G}(\mathrm{Toep}_n) \longrightarrow \mathrm{UCP}_{\mathrm{rank}=1}(\mathrm{Toep}_n) \longrightarrow 1.$$

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[Hes16] Laura Hesp. The perturbation semigroup of C^*-algebras. Master's thesis, Radboud University Nijmegen, 2016.

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