The gauge group and perturbation semigroup of an operator system

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Definition

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$. A concrete operator system is a (usually closed) linear subspace $\mathcal{E}$ of $B(\mathcal{H})$ that is closed under the involution, i.e., $x \in \mathcal{E}$ implies $x^* \in \mathcal{E}$. 
Background information about operator systems

Definition

Let $\mathcal{E}$ be an operator system, $\varphi : \mathcal{E} \to \mathcal{E}$ be a linear map, and $\varphi_n$ be the induced map $\varphi_n : M_n(\mathcal{E}) \to M_n(\mathcal{E})$.

1. The map $\varphi$ is called completely bounded if $\sup_{n>0} \|\varphi_n\| < \infty$, and we set

$$\|\varphi\|_{cb} = \sup_{n>0} \|\varphi_n\|.$$ 

2. The map $\varphi$ is called $n$–positive if $\varphi_n$ is positive, and $\varphi$ is called completely positive if $\varphi_n$ is $n$–positive for all $n > 0$.

In addition if $\varphi$ is unital we call it a unital completely positive (UCP) map.
Fact about a UCP map $\varphi$

Let $\{V_i\}_{i \leq k} \subset B(\mathcal{H})$ such that $\sum V_i^* V_i = \text{Id}$. Then the map

$$\varphi : B(\mathcal{H}) \to B(\mathcal{H})$$

$$x \mapsto \sum V_i^* x V_i$$

is a unital completely positive map.
We can embed $\mathcal{E}$ into some $C^*$-algebra $\mathcal{A}$, and then take the gauge group of $\mathcal{E}$ as the collection of unitary elements of $\mathcal{A}$ that keep $\mathcal{E}$ invariant under the unitary transformation, i.e.,

$$G(\mathcal{E}) := \{ u \in \mathcal{A} : u^* \mathcal{E} u \subset \mathcal{E} \}.$$ 

The $C^*$—algebra $\mathcal{A}$ can be taken as

1. the $C^*$-envelope,
2. the injective envelope,
3. the $C^*$-algebra $C^*(\mathcal{E})$ generated by $\mathcal{E}$.
4. else
We define the gauge group $\mathcal{G}(\mathcal{E})$ of $\mathcal{E}$ as

$$\mathcal{G}(\mathcal{E}) := \{ U \in \mathcal{U}(\mathcal{C}^*(\mathcal{E})) \mid U^* \mathcal{E} U \subset \mathcal{E} \},$$

here $\mathcal{U}(\mathcal{C}^*(\mathcal{E}))$ denotes the group of all the unitary elements in $\mathcal{C}^*(\mathcal{E})$. 
Definition

We denote by $\text{UCP}_{\text{rank}=1}(\mathcal{E})$ the collection of rank-1 unital completely positive maps, i.e.,

$$\text{UCP}_{\text{rank}=1}(\mathcal{E}) := \left\{ \varphi : \mathcal{E} \to \mathcal{E} \mid \varphi(\cdot) = V^*(\cdot)V \text{ for some } V \in B(\mathcal{H}) \text{ with } V^*V = \text{Id} \right\}.$$ 

Proposition

*There is a multiplicative map $\Psi : \mathcal{G}(\mathcal{E}) \to \text{UCP}_{\text{rank}=1}(\mathcal{E})$ defined as*

$$\Psi : U \mapsto U^*(\cdot)U, \quad U \in \mathcal{G}(\mathcal{E}).$$
Inspired by the definition of perturbation semigroups introduced in [CCvS13, NvS16, Hes16], we define the perturbation semigroup \( \text{Pert}(\mathcal{E}) \) of an operator system as follows:

**Definition**

Let \( \mathcal{E} \) be an operator system, we define the perturbation semigroup \( \text{Pert}(\mathcal{E}) \) as the collection of all the finite sums of the form

\[
\sum a_i \otimes b_i^\circ \in C^*(\mathcal{E}) \otimes C^*(\mathcal{E})^\circ
\]

satisfying the following requirements:

1. \( \sum a_i b_i = \text{Id} \),
2. \( \sum a_i \mathcal{E} b_i \subset \mathcal{E} \),
3. \( \sum a_i \otimes b_i^\circ = \sum b_i^* \otimes a_i^\circ \).

Here \( C^*(\mathcal{E})^\circ \) denotes the opposite algebra of \( C^*(\mathcal{E}) \) and \( b_i^\circ, a_i^\circ \in C^*(\mathcal{E})^\circ \).
We denote by $\text{UCBH}(\mathcal{E})$ the collection of all unital completely bounded Hermitian maps over $\mathcal{E}$, i.e.,

$$\text{UCBH}(\mathcal{E}) := \{\Psi : \mathcal{E} \to \mathcal{E} \mid \Psi(x^*) = \Psi(x)^*, \Psi(\text{Id}) = \text{Id}, \Psi \text{ is completely bounded}\}.$$ 

**Proposition ([Don21])**

There is a semigroup homomorphism $\Phi$ from $\text{Pert}(\mathcal{E})$ to $\text{UCBH}(\mathcal{E})$ defined by

$$\Phi : \text{Pert}(\mathcal{E}) \to \text{UCBH}(\mathcal{E})$$

$$\omega \mapsto \sum a_i(\cdot) b_i$$

with $\omega = \sum a_i \otimes b_i^\circ \in \text{Pert}(\mathcal{E})$. 
We denote by $\overline{\text{Pert}(\mathcal{E})}$ the closure of $\text{Pert}(\mathcal{E})$ with respect to the Haagerup tensor norm $\| \cdot \|_h$.

**Proposition ([Don21])**

Let $\mathcal{E} \subset B(\mathcal{H})$ be a unital operator system, the map $\Phi : \text{Pert}(\mathcal{E}) \to \text{UCBH}(\mathcal{E})$ can be extended to a map

$$\tilde{\Phi} : \overline{\text{Pert}(\mathcal{E})} \to \text{UCBH}(\mathcal{E}),$$

such that $\tilde{\Phi} \mid_{\text{Pert}(\mathcal{E})} = \Phi$. Moreover, if we equip $\overline{\text{Pert}(\mathcal{E})}$ and $\text{UCBH}(\mathcal{E})$ with the metric topology induced by Haagerup tensor norm $\| \cdot \|_h$ and complete bound norm $\| \cdot \|_{cb}$ respectively, the map $\tilde{\Phi}$ is contractive.
Example

Let \( \{E_{ij}\} \), \( 1 \leq i, j \leq 2 \) be the standard matrix units for \( M_2(\mathbb{C}) \). Define

\[
\text{Toep}_2 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \subset M_2(\mathbb{C}) \right\}.
\]

Take \( \omega_1, \omega_2 \in \text{Pert}(\text{Toep}_2) \) given as

\[
\omega_1 = E_{12} \otimes E_{12}^\circ + E_{21} \otimes E_{21}^\circ + E_{11} \otimes E_{11}^\circ + E_{22} \otimes E_{22}^\circ,
\]
\[
\omega_2 = (E_{12} + E_{21}) \otimes (E_{12} + E_{21})^\circ.
\]

By a direct computation we obtain that \( \Phi(\omega_1) = \Phi(\omega_2) \) on \( \text{Toep}_2 \), both give rise to the transposition map on \( \text{Toep}_2 \).

\[
\|\Phi(\omega_1)\|_{cb} = 1 < \|\omega_1\|_h = 2.
\]
Definition

We denote by $\text{Pert}^+(\mathcal{E})$ the subsemigroup of $\text{Pert}(\mathcal{E})$:

$$\text{Pert}^+(\mathcal{E}) := \{ \omega \in \text{Pert}(\mathcal{E}) \mid \omega = \sum a_i \otimes a_i^* \circ \text{ for some } a_i \in C^*(\mathcal{E}) \}.$$ 

Proposition ([Don21])

Let $\text{Pert}^+(\mathcal{E})$ be the closure of $\text{Pert}^+(\mathcal{E})$ with respect to Haagerup tensor norm. We can extend the map $\Phi : \text{Pert}^+(\mathcal{E}) \to \text{UCP}(\mathcal{E})$ to a map

$$\tilde{\Phi} : \text{Pert}^+(\mathcal{E}) \to \text{UCP}(\mathcal{E}),$$

such that $\tilde{\Phi}_{|_{\text{Pert}^+(\mathcal{E})}} = \Phi$. Moreover, we have $\|\omega\|_h = 1$ and $\|\tilde{\Phi}(\omega)\|_{cb} = 1$ for every $\omega \in \text{Pert}^+(\mathcal{E})$. 
We denote by $\text{Toep}_n$ the Toeplitz system that contains all the $n \times n$ complex Toeplitz matrices $T$ of the form

$$T := \begin{pmatrix}
    t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\
    t_{1} & t_{0} & t_{-1} & \cdots & t_{-n+2} \\
    \vdots & t_{1} & t_{0} & \ddots & \vdots \\
    t_{n-2} & \vdots & \ddots & \ddots & t_{-1} \\
    t_{n-1} & t_{n-2} & \cdots & t_{1} & t_{0}
\end{pmatrix}$$

with $t_k \in \mathbb{C}$ for $k = -n + 1, \ldots, n - 1$. 
Proposition ([Don21])

The gauge group \( \mathcal{G}(\text{Toep}_n) \) is generated by the diagonal matrices \( U_{\alpha,\beta} \) and anti-diagonal matrix \( V \) of the form

\[
U_{\alpha,\beta} = \begin{pmatrix}
\alpha & 0 & \cdots & 0 \\
0 & \beta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \overline{\alpha}^{n-2}\beta^{n-1}
\end{pmatrix}, \quad V = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 0 & 0
\end{pmatrix},
\]

here \( |\alpha| = |\beta| = 1 \).
Corollary ([Don21])

The group of $\text{UCP}_{\text{rank}=1}(\text{Toep}_n)$ is isomorphic to the semidirect product of $U(1)$ and $\mathbb{Z}_2$, and the gauge group $G(\text{Toep}_n)$ is different from $\text{UCP}_{\text{rank}=1}(\text{Toep}_n)$ by a phase factor, that is,

$$\text{UCP}_{\text{rank}=1}(\text{Toep}_n) = U(1) \rtimes \mathbb{Z}_2$$

and

$$G(\text{Toep}_n) = U(1) \rtimes (U(1) \rtimes \mathbb{Z}_2).$$

Moreover, We have the short exact sequence which is independent of $n$:

$$1 \longrightarrow U(1) \longrightarrow G(\text{Toep}_n) \longrightarrow \text{UCP}_{\text{rank}=1}(\text{Toep}_n) \longrightarrow 1.$$

