Pseudodifferential operators as generalized fixed points

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Noncommutativity in the North

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Outline

1. Generalized fixed point algebras
2. Pseudodifferential operators
3. Calculus on filtered manifolds
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1. Generalized fixed point algebras
2. Pseudodifferential operators
3. Calculus on filtered manifolds
Noncommutative version of an orbit space

Let $G$ be a locally compact group.

Commutative

- $X$ locally compact Hausdorff space
- $G \curvearrowright X$ proper group action
- then: orbit space $G \backslash X$ is locally compact and Hausdorff
Noncommutative version of an orbit space

Let $G$ be a locally compact group.

**Commutative**

- $X$ locally compact Hausdorff space
- $G \bowtie X$ proper group action
- $\rightsquigarrow G \bowtie C_0(X)$
- then: orbit space $G \setminus X$ is locally compact and Hausdorff
- $\rightsquigarrow C_0(G \setminus X)$
- Call $C_0(G \setminus X)$ the **generalized fixed point algebra** of $G \bowtie C_0(X)$
- $C_0(G \setminus X) \subset \mathcal{M}^G(C_0(X))$
Noncommutative version of an orbit space

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Noncommutative

- $A$ (noncommutative) $C^*$-algebra
- $G \curvearrowright A$ group action
- Rieffel: $G \curvearrowright A$ proper should mean: there is a generalized fixed point algebra $\text{Fix}^G(A) \subset \mathcal{M}^G(A)$ with certain properties
Construction of generalized fixed point algebras

How to get from elements of \( C_0(X) \) elements of \( \text{Fix}^G(C_0(X)) \)?

For \( f \in C_c(X) \) define \( F(Gx) := \int_G f(gx)dg \in C_0(G\backslash X) \).
Construction of generalized fixed point algebras

How to get from elements of \( C_0(X) \) elements of \( \text{Fix}^G(C_0(X)) \)?

For \( f \in C_c(X) \) define \( F(Gx) := \int_G f(gx)dg \in C_0(G \setminus X) \).

- for \( \alpha : G \curvearrowright A \): find \( \mathcal{R} \subset A \) such that
  \[
  > \int_G \alpha_g(a^* b)dg \text{ for } a, b \in \mathcal{R}
  \]
  generate a subalgebra \( \text{Fix}^G(A) \subseteq \mathcal{M}^G(A) \),

- \( \mathcal{R} \) can be completed into a Morita equivalence bimodule between \( \text{Fix}^G(A) \) and an ideal in \( A \rtimes_r G \).

- Precise conditions for \( \mathcal{R} \) described by Rieffel and Meyer [Rie04, Mey01],

- Such \( \mathcal{R} \) do not have to exist or to be unique.
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2. Pseudodifferential operators
3. Calculus on filtered manifolds
Pseudodifferential operators on $\mathbb{R}^n$

Recall that a pseudodifferential operator on $\mathbb{R}^n$ is of the form

$$Pu(x) = (2\pi)^{-n} \int e^{ix\xi} p(x, \xi) \hat{u}(\xi) \, d\xi, \quad u \in C_c^\infty(\mathbb{R}^n),$$

with a smooth function $p: T^*\mathbb{R}^n \to \mathbb{C}$ called its symbol.

**Example (Differential operators)**

$$p(x, \xi) = \sum_{|\alpha| \leq m} c_\alpha(x)(-i\xi)^\alpha \rightsquigarrow \text{differential operator } P = \sum_{|\alpha| \leq m} c_\alpha(x)\partial^\alpha.$$
Pseudodifferential operators on $\mathbb{R}^n$

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**Example (Differential operators)**

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A classical Hörmander symbol of order $m$ has an expansion

$$p(x, \xi) \sim \sum_{j=0}^\infty p_j(x, \xi)$$

where $p_j$ is $(m - j)$-homogeneous with respect to $\alpha: \mathbb{R}^n \sim T^*\mathbb{R}^n$:

$$\alpha_\lambda(x, \xi) = (x, \lambda \xi).$$
Pseudodifferential extension

Let $M$ be a closed manifold.

- denote by $\Psi^m(M)$ the classical pseudodifferential operators of order $m$,
- mapping $P \in \Psi^m(M)$ to its principal symbol $p_0 \in C^\infty(S^*M)$ is well-defined and induces a short exact sequence

$$\psi^{m-1}(M) \hookrightarrow \psi^m(M) \xrightarrow{\sigma_m} C^\infty(S^*M),$$
Pseudodifferential extension

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$$\Psi^{m-1}(M) \hookrightarrow \Psi^m(M) \xrightarrow{\sigma^m} C^\infty(S^*M),$$

- for $m = 0$, this can be completed into a SES of $C^*$-algebras

$$K(L^2(M)) \hookrightarrow C^*(\Psi^0(M)) \xrightarrow{\sigma_0} C(S^*M), \quad (1)$$

- implies: $P \in \Psi^0(M)$ is Fredholm $\iff$ $P$ is elliptic ($\sigma_0(P)$ is invertible).
- **Observation:** $(1)$ is a sequence of generalized fixed point algebras!
Connes' tangent groupoid

\[ \mathcal{T}M = M \times M \times (0, \infty) \cup TM \times \{0\} \]

zoom action \( \alpha : \mathbb{R}_{>0} \times TM \)

\[ \lambda \cdot (x, v, 0) = (x, \lambda v, 0), \quad \forall v \in T_x M, \]

\[ \lambda \cdot (x, y, t) = (x, y, \lambda^t t), \quad t > 0, \quad (x, y) \in M \times M. \]
Connes' tangent groupoid

\[ TM = M \times M \times (0, \infty) \cup TM \times 0^3 \]

Zoom action \( \alpha : \mathbb{R}_{>0} \to TM \)

\[ \lambda \cdot (x, v, 0) = (x, \lambda v, 0), \quad \forall \lambda \in \mathbb{R}_{>0}, \]

\[ \lambda \cdot (x, y, t) = (x, y, \lambda t), \quad t > 0, \quad (x, y) \in M \times M. \]

Groupoid C*-algebra

\[ C_0((0, \infty), \mathbb{R}(L^2 M)) \hookrightarrow C^*(TM) \xrightarrow{ev_0} C^*(TM) \]

\[ \ell^2 \uparrow \]

\[ C_0(T^* M) \]

Induced action \( \alpha : \mathbb{R}_{>0} \to C^*(TM) \)

At \( t = 0 \): \( \alpha \) corresponds under \( \uparrow \)

to

\[ \lambda \cdot f(x, \xi) = f(x, \lambda \xi) \]

(not proper: \( \xi = 0 \))
Connes' tangent groupoid

\[ TM = M \times M \times (0, \infty) \cup TM \times \{0\} \]

zoom action \( \alpha : \mathbb{R}_{>0} \triangleright TM \)

\[ \lambda \cdot (x, y, t) = (x, y, \lambda \cdot t), \quad t > 0, \quad (x, y) \in M \times M. \]

Groupoid C*-algebra

\[ \begin{array}{c}
C_0((0, \infty), \mathcal{K}(L^2(M))) \hookrightarrow C^*(TM) \\
\xrightarrow{\text{ev}_0} \quad C^*(TM) \\
\xrightarrow{\text{induced action} \ \alpha : \mathbb{R}_{>0} \triangleright C^*(TM)} \\
\text{at } t = 0: \ \alpha \text{ corresponds under } \wedge \text{ to } \lambda : f(x, \xi) = f(x, \lambda \xi) \\
\text{(not proper: } \xi = 0) \end{array} \]

restrict to ideals to make \( \mathbb{R}_{>0} \)-actions "proper"

\[ J = \{ f \in C^*(TM) : \widehat{f}_{0,x}(0) = 0 \ \forall x \in M \} \]

\[ J_0 = \text{ev}_0(J) \]

\[ C_0((0, \infty), \mathcal{K}(L^2(M))) \hookrightarrow J \xrightarrow{\text{ev}_0} J_0 \]

\( \mathbb{R}_{>0} \)-invariant ideals

"proper" \( \mathbb{R}_{>0} \)-actions
Connes' tangent groupoid

\[ TM = M \times M \times (0, \infty) \cup TM \times \{0\} \]

**zoom action** \( \alpha : R_{>0} \triangleright TM \)

\[ \lambda \cdot (x, v, 0) = (x, \lambda v, 0), \quad \forall \in \mathcal{T} M, \]

\[ \lambda \cdot (x, y, t) = (x, y, \lambda^{-1} t), \quad t > 0, \quad (x, y) \in M \times M. \]

**Groupoid C*-algebra**

\[ C_0((0, \infty), \mathcal{B}(L^2 M)) \to C^*(TM) \overset{\text{ev}_0}{\to} C^*(TM) \]

\[ \overset{\text{L}^2 \backslash}{\overset{\text{C}_0(T^*M)}{\to}} \]

**induced action** \( \alpha : R_{>0} \triangleright C^*(TM) \)

at \( t = 0 \): \( \alpha \) corresponds under \( ^\triangleleft \)

\[ \lambda \cdot f(x, z) = f(x, \lambda z) \]

(not proper: \( z = 0 \))

**restrict to ideals to make** \( R_{>0} \)-**actions "proper"**

\[ J = \{ f \in C^*(TM) : \widehat{\text{fo}}_x (0) = 0 \quad \forall x \in M \}, \quad J_0 = \text{ev}_0 (J) \]

**\( R_{>0} \)-invariant ideals**

\[ C_0((0, \infty), \mathcal{B}(L^2 M)) \to J \overset{\text{ev}_0}{\to} J_0 \]

**"proper"** \( R_{>0} \)-**actions**

**Generalized fixed point algebra**

\[ \text{Fix}^{R_{>0}} (C_0((0, \infty), \mathcal{B}(L^2 M))) \overset{\text{L}_2}{\to} \text{Fix}^{R_{>0}} (J) \overset{\text{L}_2}{\to} \text{Fix}^{R_{>0}} (J_0) \]

\[ \overset{\text{L}_2}{\overset{\text{K}(L^2 M)}{\to}} \overset{\text{C}^*(\Psi^0(M))}{\to} \overset{\text{C}_0(S^*M)}{\to} \]
Pseudodifferential operators as generalized fixed points

Similar constructions to obtain pseudodifferential extensions for

- classical Shubin calculus on $\mathbb{R}^n$ (use $\lambda \cdot (x, \xi) = (\lambda x, \lambda \xi)$ on $T^*\mathbb{R}^n$),
- classical SG-calculus on $\mathbb{R}^n$,
- filtered manifolds and graded Lie groups.
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Motivation

• Usual pseudodifferential calculus: elliptic differential operators are hypoelliptic and on closed manifolds Fredholm,
• but not all Fredholm or hypoelliptic differential operators are elliptic.

Examples

• $X^2 + Y^2 + i\mu Z$ for $\mu \notin 2\mathbb{Z} + 1$ on the Heisenberg group,
• heat operator $\Delta + \partial_t$.

• Can this be understood in terms of a suitable pseudodifferential calculus?

• Idea: adjust notion of order and use nilpotent groups as models. [Fol77, RS76, Mel82, Tay84, CGGP92, Pon08, vEY19, DH17] and others
Filtered manifolds and new notion of order

Definition

A **filtered manifold** is a smooth manifold $M$ with a filtration of subbundles $0 = H^0 \subseteq H^1 \subseteq \ldots \subseteq H^r = TM$ such that $[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^{i+j})$. Here, set $H^k = TM$ for all $k > r$. 
Filtered manifolds and new notion of order

Definition

A filtered manifold is a smooth manifold $M$ with a filtration of subbundles $0 = H^0 \subseteq H^1 \subseteq \ldots \subseteq H^r = TM$ such that $[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^{i+j})$. Here, set $H^k = TM$ for all $k > r$.

New notion of order:

- $X \in \Gamma^\infty(H^k) \setminus \Gamma^\infty(H^{k-1})$ defines a differential operator of order $k$,
- $DO^m_H(M)$: differential operators of new order $\leq m$.

Examples

- every smooth manifold for $r = 1$, contact manifolds, graded Lie groups, equiregular Carnot-Carathéodory manifolds, ...
Principal part acting on osculating groups

- Lie bracket induces a nilpotent Lie algebra structure on each fibre of

\[ t_H M := \bigoplus_{i=1}^{r} \frac{H^i}{H^{i-1}}, \]

- the fibres \( g_x \) integrate to graded Lie groups \( G_x \) for \( x \in M \) called the osculating groups,
- not all osculating groups need to be isomorphic,
Principal part acting on osculating groups

- Lie bracket induces a nilpotent Lie algebra structure on each fibre of
  \[ t_H M := \bigoplus_{i=1}^{r} H^i / H^{i-1}, \]

- the fibres \( g_x \) integrate to graded Lie groups \( G_x \) for \( x \in M \) called the osculating groups,

- not all osculating groups need to be isomorphic,

- well-defined principal part map \( \sigma_m : DO_H^m(M) \to U(t_H M) \),

- \( \sigma_m(P)_x \) is a left-invariant differential operator on \( G_x \) for each \( x \in M \).

Step \( r = 1 \) filtration

- all osculating groups are \( G_x = T_x M \cong (\mathbb{R}^n, +) \),

- \( \sigma_m(P)_x \): taking the highest order part and freezing coefficients at \( x \),

- \( p_0(x, \xi) = \sigma_m(P)_x(\xi) \) principal symbol.
Rockland condition as a replacement for ellipticity

- Step $r = 1$ filtration: $P$ is elliptic if $p(x, \xi) = \sigma_m(P)_x(\xi)$ is invertible for all $x \in M$ and $\xi \neq 0$,
- when $G_x$ is not Abelian, we can also take Fourier transform: in terms of representations $\pi : G_x \to U(\mathcal{H}_\pi)$,
- every $\pi \in \hat{G}_x$ induces an infinitesimal representation $d\pi$ of $U(g_x)$ as possibly unbounded operators on $\mathcal{H}_\pi$.

**Definition**

An operator $P \in DO^m_H(M)$ satisfies the Rockland condition if for all $x \in M$ and all $\pi \in \hat{G}_x \setminus \{\pi_{\text{triv}}\}$ the operator $d\pi(\sigma_m(P)_x)$ is injective on $\mathcal{H}_\pi$. 
Generalized fixed point algebra construction

- [vEY19, CP19] defined a tangent groupoid $\mathbb{T}_H M$ for filtered manifolds

$$\mathbb{T}_H M = M \times M \times (0, \infty) \cup (G_x)_{x \in M} \times \{0\} \Rightarrow M \times [0, \infty),$$

- zoom action $\mathbb{R}_{>0} \curvearrowright \mathbb{T}_H M$ encodes the new notion of order.

- Generalized fixed point algebra construction gives $K(L^2(M)) = \text{Fix}_{\mathbb{R}_{>0}}(J)$ $\text{Fix}_{\mathbb{R}_{>0}}(J_0)$. 

$\sigma_0$ Eske Ewert
Generalized fixed point algebra construction

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$$\mathbb{T}_H M = M \times M \times (0, \infty) \cup (G_x)_{x \in M} \times \{0\} \supseteq M \times [0, \infty),$$

• zoom action $\mathbb{R}_> \sim \mathbb{T}_H M$ encodes the new notion of order.

Observation for $G_x = (\mathbb{R}^n, +)$ and $f \in C^\infty_c(\mathbb{R}^n)$

$$\hat{f}(0) = 0 \iff \int_{\mathbb{R}^n} f(x) dx = 0$$
Generalized fixed point algebra construction

- [vEY19, CP19] defined a tangent groupoid $T_HM$ for filtered manifolds
  \[ T_HM = M \times M \times (0, \infty) \cup (G_x)_{x \in M} \times \{0\} \Rightarrow M \times [0, \infty), \]
  
- zoom action $\mathbb{R}_{>0} \lhd T_HM$ encodes the new notion of order.

**Observation for $G_x = (\mathbb{R}^n, +)$ and $f \in C^\infty_c(\mathbb{R}^n)$**

\[ \tilde{f}(0) = 0 \Leftrightarrow \int_{\mathbb{R}^n} f(x)\pi_{\text{triv}}(x)dx = 0 \Leftrightarrow f \in \ker(\pi_{\text{triv}} : C^*(\mathbb{R}^n) \to \mathbb{C}) \]
Generalized fixed point algebra construction

- [vEY19, CP19] defined a tangent groupoid $T_H M$ for filtered manifolds
  \[ T_H M = M \times M \times (0, \infty) \cup (G_x)_{x \in M} \times \{0\} \leadsto M \times [0, \infty), \]
- zoom action $\mathbb{R}_{>0} \curvearrowright T_H M$ encodes the new notion of order.

**Observation for $G_x = (\mathbb{R}^n, +)$ and $f \in C_c^\infty(\mathbb{R}^n)$**

\[ \hat{f}(0) = 0 \iff \int_{\mathbb{R}^n} f(x)\pi_{\text{triv}}(x) dx = 0 \iff f \in \ker(\pi_{\text{triv}} : C^*(\mathbb{R}^n) \to \mathbb{C}) \]

- Take ideal
  \[ J := \{ f \in C^*(T_H M) : f_{0, x} \in \ker(\pi_{\text{triv}} : C^*(G_x) \to \mathbb{C}) \text{ for all } x \in M \}, \]
- Generalized fixed point algebra construction gives
  \[ \mathbb{K}(L^2(M)) \hookrightarrow \text{Fix}^{\mathbb{R}_{>0}}(J) \xrightarrow{\sigma_0} \text{Fix}^{\mathbb{R}_{>0}}(J_0). \]
Results (E.)

Theorem

\[ \mathbb{K}(L^2(M)) \leftrightarrow \text{Fix}^{\mathbb{R}^0}(J) \xrightarrow{\sigma_0} \text{Fix}^{\mathbb{R}^0}(J_0) \]

is the $C^*$-completion of the order zero pseudodifferential extension by

- van Erp and Yuncken for filtered manifolds,
- Fermanian-Kammerer–Fischer–Ruzhansky for graded Lie groups.

For $M$ closed: $P \in \text{Fix}^{\mathbb{R}^0}(J)$ Fredholm $\iff \sigma_0(P) \in \text{Fix}^{\mathbb{R}^0}(J_0)$ invertible.

Properties of generalized fixed point algebras allow to show:

- spectrum of $\text{Fix}^{\mathbb{R}^0}(J_0)$ is $M \times (\hat{G} \setminus \{\pi_{\text{triv}}\})/\mathbb{R}^0$,
- $P \in \psi^0_H(M)$ is Fredholm $\iff P$ and $P^*$ satisfy the Rockland condition,
- $\text{Fix}^{\mathbb{R}^0}(J_0)$ is $KK$-equivalent to $C(S^*M)$. 
Thank you!
References


