$C^*$-algebras associated to homeomorphisms twisted by vector bundles over finite dimensional spaces

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based on joint work with Adamo, Archey, Georgescu, Jeong, Strung and Viola and ongoing project with Adamo and Strung
**Definition.** Let $A$ be $C^*$-algebra. A $C^*$-correspondence over $A$ is a right Hilbert $A$-module $E$ equipped with a $^*$-homomorphism,

$$
\varphi_E : A \to \mathcal{L}(E),
$$

called the **structure map**.

Let $E$ be a right Hilbert $A$-module and left Hilbert $A$-module. We say that $E$ is an $A$-Hilbert bimodule if

$$
\xi \langle \eta, \zeta \rangle_E = E \langle \xi, \eta \rangle \zeta, \quad \xi, \eta, \zeta \in E,
$$

where $\langle \cdot, \cdot \rangle_E$ denotes the right inner product and $E \langle \cdot, \cdot \rangle$ the left inner product.
**Definition.** Let $A$ be a $C^*$-algebra. A $C^*$-correspondence over $A$ is a right Hilbert $A$-module $\mathcal{E}$ equipped with a $^*$-homomorphism,

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\varphi_{\mathcal{E}} : A \to \mathcal{L}(\mathcal{E}),
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called the *structure map*.

The map $\varphi_{\mathcal{E}}$ gives $\mathcal{E}$ a left $A$-module structure, but not necessarily a left Hilbert $A$-module structure, as there need not be a left $A$-valued inner product on $\mathcal{E}$. 
Definition. Let $A$ be $C^*$-algebra. A $C^*$-correspondence over $A$ is a right Hilbert $A$-module $E$ equipped with a $^*$-homomorphism,

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Let $E$ be a right Hilbert $A$-module and left Hilbert $A$-module. We say that $E$ is an $A$-Hilbert bimodule if

$$\xi \langle \eta, \zeta \rangle_E = \varepsilon \langle \xi, \eta \rangle \zeta, \quad \xi, \eta, \zeta \in E,$$

where $\langle \cdot, \cdot \rangle_E$ denotes the right inner product and $\varepsilon \langle \cdot, \cdot \rangle$ the left inner product.
Cuntz-Pimsner algebras

**Definition.** Let $A$ and $B$ be $\mathcal{C}^*$-algebras and let $\mathcal{E}$ be a $\mathcal{C}^*$-correspondence over $A$ with structure map $\varphi_\mathcal{E} : A \to \mathcal{L}(\mathcal{E})$. A representation $(\pi, \tau)$ of $\mathcal{E}$ on $B$ consists of a $\ast$-homomorphism $\pi : A \to B$ and a linear map $\tau : \mathcal{E} \to B$ satisfying

1. $\pi(\langle \xi, \eta \rangle_\mathcal{E}) = \tau(\xi)^\ast \tau(\eta)$, for every $\xi, \eta \in \mathcal{E}$;
2. $\pi(a)\tau(\xi) = \tau(\varphi_\mathcal{E}(a)\xi)$, for every $\xi \in \mathcal{E}, a \in A$.

Let $\psi_\tau : \mathcal{K}(\mathcal{E}) \to B$ be the $\ast$-homomorphism defined on rank one operators by

$$\psi_\tau(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^\ast, \quad \xi, \eta \in \mathcal{E}.$$ 

We say that the representation $(\pi, \tau)$ is **covariant** if in addition

- $\pi(a) = \psi_\tau(\varphi_\mathcal{E}(a))$ for every $a \in J_\mathcal{E}$, where

$$J_\mathcal{E} := \varphi_\mathcal{E}^{-1}(\mathcal{K}(\mathcal{E})) \cap (\ker \varphi_\mathcal{E})^\perp$$
**Definition.** Let $A$ be a $C^*$-algebra and let $\mathcal{E}$ be a $C^*$-correspondence over $A$. The *Cuntz–Pimsner algebra of $\mathcal{E}$ over $A$*, denoted $\mathcal{O}_A(\mathcal{E})$ (or simply $\mathcal{O}(\mathcal{E})$ if the $C^*$-algebra $A$ is understood) is the $C^*$-algebra generated by the universal covariant representation of $\mathcal{E}$. By universality, we mean that the universal covariant representation $(\pi_u, \tau_u)$ satisfies the following: for any covariant representation $(\pi, \tau)$ of $\mathcal{E}$ there exists a surjective linear map $\psi: C^*(\pi_u, \tau_u) \to C^*(\pi, \tau)$ such that $\pi = \psi \circ \pi_u$ and $\tau = \psi \circ \tau_u$. 
**Definition.** Let $A$ be a $\mathcal{C}^*$-algebra and let $\mathcal{E}$ be a $\mathcal{C}^*$-correspondence over $A$. The *Cuntz–Pimsner algebra of $\mathcal{E}$ over $A$*, denoted $\mathcal{O}_A(\mathcal{E})$ (or simply $\mathcal{O}(\mathcal{E})$ if the $\mathcal{C}^*$-algebra $A$ is understood) is the $\mathcal{C}^*$-algebra generated by the universal covariant representation of $\mathcal{E}$.

By universality, we mean that the universal covariant representation $(\pi_u, \tau_u)$ satisfies the following: for any covariant representation $(\pi, \tau)$ of $\mathcal{E}$ there exists a surjective linear map $\psi: \mathcal{C}^*(\pi_u, \tau_u) \to \mathcal{C}^*(\pi, \tau)$ such $\pi = \psi \circ \pi_u$ and $\tau = \psi \circ \tau_u$. 

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$\mathcal{C}^*$-algebras associated to homeomorphisms twisted by vector bundles
Serre-Swan theorem says that if $X$ is a compact metric space and $\mathcal{E}$ is an algebraically finitely generated right $C(X)$-module, then there exists a vector bundle $\mathcal{V} = [V, p, X]$ such that $\mathcal{E} \cong \Gamma(\mathcal{V})$ where $\Gamma(\mathcal{V})$ denotes the $C(X)$-module of continuous sections of $\mathcal{V}$. 
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We denote a vector bundle over a locally compact metric space $X$ by $\mathcal{V} = [V, p, X]$, where $p : V \to X$ is a continuous surjective map and for every $x \in X$, the fibre $p^{-1}(x) \cong \mathbb{C}^{n_x}$ for some $n_x \in \mathbb{Z}_{\geq 0}$. 
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A chart for a vector bundle $\mathcal{V} = [V, p, X]$ is an open subset of $X$ together with an isomorphism $h : U \times \mathbb{C}^{nu} \to \mathcal{V}|_U$ where $\mathbb{C}^{nu} \cong p^{-1}(x)$ for any $x \in U$. 
Serre-Swan theorem says that if \( X \) is a compact metric space and \( \mathcal{E} \) is an algebraically finitely generated right \( C(X) \)-module, then there exists a vector bundle \( \mathcal{V} = [V, p, X] \) such that \( \mathcal{E} \cong \Gamma(\mathcal{V}) \) where \( \Gamma(\mathcal{V}) \) denotes the \( C(X) \)-module of continuous sections of \( \mathcal{V} \).

We denote a vector bundle over a locally compact metric space \( X \) by \( \mathcal{V} = [V, p, X] \), where \( p : V \to X \) is a continuous surjective map and for every \( x \in X \), the fibre \( p^{-1}(x) \cong \mathbb{C}^{n_x} \) for some \( n_x \in \mathbb{Z}_{\geq 0} \).

A chart for a vector bundle \( \mathcal{V} = [V, p, X] \) is an open subset of \( X \) together with an isomorphism \( h : U \times \mathbb{C}^{n_U} \to \mathcal{V}|_U \) where \( \mathbb{C}^{n_U} \cong p^{-1}(x) \) for any \( x \in U \).

An atlas for \( \mathcal{V} \) is a family of charts \( \{h_i : U_i \times \mathbb{C}^{n_i} \to \mathcal{V}|_{U_i}\}_{i \in I} \) such that the \( U_i \) cover \( X \).
The right $C(X)$-module $\Gamma(\mathcal{V})$ admits a right $C(X)$-valued inner product defined as follows.

Let $(\{U_i, h_i\})_{i=1}^N$ be an atlas for $\mathcal{V}$ and $\gamma_1, \ldots, \gamma_N$ be a partition of unity subordinate to $U_1, \ldots, U_N$. Define

$$\langle \xi, \eta \rangle_{\Gamma(\mathcal{V})}(x) := \sum_{j=1}^N \gamma_j(x) \langle h_j^{-1}(\xi(x)), h_j^{-1}(\eta(x)) \rangle_{\mathbb{C}^n_j},$$

which makes $\Gamma(\mathcal{V})$ into a right Hilbert $C(X)$-module.
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**Fact.** Let $X$ be an infinite compact metric space. Suppose that $\mathcal{E}$ is a finitely generated projective right Hilbert $C(X)$-module. Then there exists a vector bundle $\mathcal{V} = [V, \rho, X]$ and a unitary isomorphism $U : \mathcal{E} \to \Gamma(\mathcal{V})$, where $\Gamma(\mathcal{V})$ is equipped with an inner product as defined in (1) with respect to any choice of atlas for $\mathcal{V}$. 
Examples.

- Let $\mathcal{V} = [X \times \mathbb{C}^n, p, X]$ be a trivial bundle with constant rank $n > 1$, and let $\varphi : C(X) \to \mathcal{L}(\Gamma(\mathcal{V}))$ be given by $\varphi(f)(\xi) = \xi f$. Then $\mathcal{O}(\Gamma(\mathcal{V})) \cong C(X, \mathcal{O}_n)$.

- Let $\mathcal{E}$ be the right Hilbert $C(X)$-module of sections associated to a trivial line bundle $\mathcal{V} = [X \times \mathbb{C}, p, X]$ and let $\alpha : X \to X$ be a homeomorphism. Define $\varphi : C(X) \to \mathcal{L}(\Gamma(\mathcal{V}))$ by $\varphi(f)(\xi) = \xi f \circ \alpha$, then $\mathcal{O}(\mathcal{E}) \cong C(X) \rtimes_{\alpha} \mathbb{Z}$.

- Let $X$ be a compact metric space, $\mathcal{V}$ a vector bundle over $X$ and $\alpha : X \to X$ a homeomorphism. Denote by $\Gamma(\mathcal{V}, \alpha)$ the $C^*$-correspondence which has right Hilbert $C(X)$-module structure given by $\Gamma(\mathcal{V})$ and structure map $\varphi : C(X) \to \mathcal{K}(\Gamma(\mathcal{V}))$ by $\varphi(f)\xi = \xi f \circ \alpha$. 

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We have the following characterisation of Hilbert $C(X)$-bimodules which are finitely generated projective as right Hilbert $C(X)$-modules.

**Proposition.** Let $\mathcal{E}$ be a non-zero Hilbert $C(X)$-bimodule which is finitely generated projective as a right Hilbert $C(X)$-module. Then there exist a compact metric space $Y \cong X$, line bundle $\mathcal{V} = [V, p, X]$ and homeomorphism $\alpha : X \rightarrow Y$ such that

- $\mathcal{E}_{C(X)} \cong \Gamma(\mathcal{V})$;
- $c(X)\mathcal{E} \cong \Gamma((\alpha^{-1})^*\mathcal{V})$;
- $f\xi = \xi f \circ \alpha$ for every $f \in C(X)$ and every $\xi \in \mathcal{E}$. 

If $\mathcal{E}$ is left full, then we may take $X = Y$ and $\mathcal{E} \cong \Gamma(\mathcal{V}, \alpha)$. 

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If $\mathcal{E}$ is left full, then we may take $X = Y$ and $\mathcal{E} \cong \Gamma(\mathcal{V}, \alpha)$. 
A homeomorphism $\alpha : X \to X$ is \textit{minimal} if, whenever $E \subset X$ is a closed subset such that $\alpha(E) \subset E$, then $E \in \{\emptyset, X\}$.
A homeomorphism $\alpha : X \to X$ is minimal if, whenever $E \subset X$ is a closed subset such that $\alpha(E) \subset E$, then $E \in \{\emptyset, X\}$, or equivalently, for every $x \in X$, the orbit of $x$,

$orb(x) := \{\alpha^n(x) \mid n \in \mathbb{Z}\}$ is dense in $X$. 

Theorem

Let $X$ be a compact metric space, $V$ a vector bundle over $X$ and $\alpha : X \to X$ a homeomorphism. Then $O(\Gamma(\mathcal{Y}, \alpha))$ is simple if and only if $\alpha$ is minimal.
Simplicity of \( \mathcal{O}(\Gamma(\mathcal{V}, \alpha)) \)

A homeomorphism \( \alpha : X \to X \) is *minimal* if, whenever \( E \subset X \) is a closed subset such that \( \alpha(E) \subset E \), then \( E \in \{\emptyset, X\} \),

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**Theorem**  Let \( X \) be a compact metric space, \( \mathcal{V} \) a vector bundle over \( X \) and \( \alpha : X \to X \) a homeomorphism. Then \( \mathcal{O}(\Gamma(\mathcal{V}, \alpha)) \) is simple if and only if \( \alpha \) is minimal.
**Theorem** Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ where $\mathcal{V} = [V, p, X]$ is a vector bundle and $\alpha : X \to X$ is a homeomorphism. Then $T(\mathcal{O}(\mathcal{E})) \neq \emptyset$ if and only if $\mathcal{V}$ is a line bundle.
Trace space of $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$

**Theorem** Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ where $\mathcal{V} = [V, p, X]$ is a vector bundle and $\alpha : X \to X$ is a homeomorphism. Then $T(\mathcal{O}(\mathcal{E})) \neq \emptyset$ if and only if $\mathcal{V}$ is a line bundle.

The gauge action gives us an associated conditional expectation onto the fixed point $C^*$-subalgebra given by

$$\Phi : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})^\sigma, \quad a \mapsto \int_T \sigma_z(a)dz. \quad (2)$$

Let $\mathcal{V}$ be a line bundle. Let $\mu$ be an $\alpha$-invariant probability measure on $X$, and let

$$\tau_{\mu}(f) = \int_X fd\mu,$$

which is evidently a state. Then $\tau_{\mu} \circ \Phi$ is tracial.
Trace space of $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$

**Theorem** Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ where $\mathcal{V} = [V, p, X]$ is a vector bundle and $\alpha : X \to X$ is a homeomorphism. Then $T(\mathcal{O}(\mathcal{E})) \neq \emptyset$ if and only if $\mathcal{V}$ is a line bundle.

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Let $\mathcal{V}$ be a line bundle. Let $\mu$ be an $\alpha$-invariant probability measure on $X$, and let

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\tau_\mu(f) = \int_X f d\mu,
$$

which is evidently a state. Then $\tau_\mu \circ \Phi$ is tracial. Indeed, every tracial state arises in this way.
**Theorem** Let $X$ be an infinite compact metric space, $\mathcal{V} = [V, p, X]$ a line bundle, and $\alpha : X \to X$ an aperiodic homeomorphism. Let $\mathcal{E} := \Gamma(\mathcal{V}, \alpha)$.

Then there are affine homeomorphisms

$$T(\mathcal{O}(\mathcal{E})) \cong T(C(X) \rtimes_{\alpha} \mathbb{Z}) \cong M^1(X, \alpha),$$

where $M^1(X, \alpha)$ denotes the space of $\alpha$-invariant Borel probability measures.
Theorem. Let $X$ be an infinite compact metric space, $\mathcal{V} = [V, p, X]$ a line bundle, and $\alpha : X \to X$ an aperiodic homeomorphism. Let $\mathcal{E} := \Gamma(\mathcal{V}, \alpha)$. Then there are affine homeomorphisms

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where $M^1(X, \alpha)$ denotes the space of $\alpha$-invariant Borel probability measures.

Corollary. Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$ where $\mathcal{V} = [V, p, X]$ is a line bundle and $\alpha : X \to X$ is a minimal homeomorphism. Then $\mathcal{O}(\mathcal{E})$ is stably finite.
Theorem (Elliott, Gong, Lin, Niu, Tikuisis, Winter, White,...) Let $A$ and $B$ be simple, separable, unital with finite nuclear dimension and which satisfy the UCT. Then $A \cong B$ if and only if $Ell(A) \cong Ell(B)$.
Theorem (Elliott, Gong, Lin, Niu, Tikuisis, Winter, White,...) Let $A$ and $B$ be simple, separable, unital with finite nuclear dimension and which satisfy the UCT. Then $A \cong B$ if and only if $\text{Ell}(A) \cong \text{Ell}(B)$.

We have already showed that when $O(\Gamma(\mathcal{V}, \alpha))$ is simple and Katsura proved that $O(\Gamma(\mathcal{V}, \alpha))$ satisfy the UCT. So when does it have finite nuclear dimension?
Theorem Let $X$ be an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha: X \to X$ an aperiodic homeomorphism. Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$, then $O(\mathcal{E})$ has finite nuclear dimension.
Theorem Let $X$ be an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha : X \to X$ an aperiodic homeomorphism. Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$, then $\mathcal{O}(\mathcal{E})$ has finite nuclear dimension. If $\alpha$ is minimal, then the nuclear dimension is one.
**Theorem** Let $X$ be an infinite compact metric space with $\dim(X) < \infty$, $\mathcal{V} = [V, p, X]$ a vector bundle, and $\alpha : X \to X$ an aperiodic homeomorphism. Let $\mathcal{E} = \Gamma(\mathcal{V}, \alpha)$, then $\mathcal{O}(\mathcal{E})$ has finite nuclear dimension. If $\alpha$ is minimal, then the nuclear dimension is one.

**Theorem** Let $X, Y$ be compact metric spaces with finite dimensional, $\alpha : X \to X$ and $\beta : Y \to Y$ be minimal homeomorphisms and let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles over $X$ and $Y$, respectively. Then $\mathcal{O}_{C(X)}(\Gamma(\mathcal{V}, \alpha)) \cong \mathcal{O}_{C(Y)}(\Gamma(\mathcal{W}, \beta))$ if and only if

$$\text{Ell}(\mathcal{O}_{C(X)}(\Gamma(\mathcal{V}, \alpha))) \cong \text{Ell}(\mathcal{O}_{C(Y)}(\Gamma(\mathcal{W}, \beta))).$$
**Corollary** Let where $\alpha: X \to X$ is a minimal homeomorphism and $\dim(X) < \infty$. Then $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ is stable rank one if and only if $\mathcal{V}$ is a line bundle. Otherwise it is purely infinite.
Let $\mathcal{V}$ be a line bundle over a compact metric space and $\alpha: X \rightarrow X$ be a homeomorphism.

Theorem. $\mathcal{C}(X)$ is a maximal abelian subalgebra of $\mathcal{O}(\Gamma(\mathcal{V},\alpha))$ if and only if $\alpha$ is topologically free (that is, the set of aperiodic points is dense).
Let $\mathcal{V}$ be a line bundle over compact metric space and $\alpha: X \to X$ be a homeomorphism.

When does $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ have a Cartan subalgebra?
Let $\mathcal{V}$ is a line bundle over compact metric space and $\alpha: X \rightarrow X$ be a homeomorphism.

When does $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ have a Cartan subalgebra?

Or in particular, when $C(X)$ is a Cartan subalgebra of $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$?

**Theorem.** $C(X)$ is a maximal abelian subalgebra of $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ if and only if $\alpha$ is toplogically free (that is the set of aperiodic points is dense).
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**Theorem.** $C(X)$ is a maximal abelian subalgebra of $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ if and only if $\alpha$ is topologically free (that is the set of aperiodic points is dense).
Theorem  The following conditions are equivalent:

(a) $C(X)$ is a Cartan subalgebra of $\mathcal{O}(\Gamma(\mathcal{V}, \alpha))$;
(b) $\alpha$ is topologically free;
(c) If ideal $I \triangleleft \mathcal{O}(\Gamma(\mathcal{V}, \alpha))$ has zero intersection with $C(X)$, then $I = 0$. 
**Theorem** The following conditions are equivalent:

(a) $C(X)$ is a Cartan subalgebra of $\mathcal{O}(\Gamma(V, \alpha))$;

(b) \( \alpha \) is topologically free;

(c) If ideal \( I \triangleleft \mathcal{O}(\Gamma(V, \alpha)) \) has zero intersection with \( C(X) \), then \( I = 0 \).

**Question** What is the role of the line bundle \( V \) in the description of the Cartan pair of \( (\mathcal{O}(\Gamma(V, \alpha)), C(X)) \)?
Renault constructed a bijection between isomorphism classes of twists over topologically principal, second countable etale groupoids and isomorphism classes of Cartan pairs of separable $C^*$-algebras.
Renault constructed a bijection between isomorphism classes of twists over topologically principal, second countable etale groupoids and isomorphism classes of Cartan pairs of separable $C^*$-algebras.

Let $B$ be a Cartan subalgebra of separable $C^*$-algebra $A$, Renault, based on the work of Kumjian, constructed Weyl groupoid $\mathcal{G}_{(A,B)}$ and Weyl twist $\Sigma_{(A,B)}$ over $\mathcal{G}_{(A,B)}$ such that $(C(\mathcal{G}_{(A,B)}^{(0)}), C^*_r(\mathcal{G}_{(A,B)}; \Sigma_{(A,B)}))$ is a Cartan pair and isomorphic to $(B, A)$. 

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Theorem. The Weyl groupoid associated to the Cartan pair $(\mathcal{O}(\Gamma(\mathcal{V}, \alpha)), \mathcal{C}(X))$ is isomorphic to $X \times_\alpha \mathbb{Z}$.

For every line bundle $\mathcal{V}$ over $X$, one can associate a principal circle bundle $P\mathcal{V}$ in a "canonical way". Define $j_\alpha: X \times_\alpha \mathbb{Z} \to X$ by $j_\alpha(\alpha k(x), k, x) \mapsto x$, then Deaconu-Kumjian-Muhly showed that $j_\alpha^*(P\mathcal{V})$ defines a twist on $X \times_\alpha \mathbb{Z}$. Let denote this twist by $\Sigma_{\mathcal{V}, \alpha}$. 
**Theorem.** The Weyl groupoid associated to the Cartan pair $(\mathcal{O}(\Gamma(\mathcal{V}, \alpha)), C(X))$ is isomorphic to $X \times_\alpha \mathbb{Z}$.

For every line bundle $\mathcal{V}$ over $X$, one can associate a principal circle bundle $P_{\mathcal{V}}$ in a "canonical way".

Define $j_\alpha: X \times_\alpha \mathbb{Z} \to X$ by $j_\alpha(\alpha k(x), k, x) \mapsto x$, then Deaconu-Kumjian-Muhly showed that $j_\alpha^*\alpha(P_{\mathcal{V}})$ defines a twist on $X \times_\alpha \mathbb{Z}$. Let denote this twist by $\Sigma_{\mathcal{V}, \alpha}$.
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For every line bundle $\mathcal{V}$ over $X$, one can associate a principal circle bundle $\mathcal{P}_\mathcal{V}$ in a ”canonical way”.

Define $j_\alpha : X \times_\alpha \mathbb{Z} \to X$ by $j_\alpha(\alpha^k(x), k, x) \mapsto x$, then Deaconu-Kumjian-Muhly showed that $j_\alpha^*(\mathcal{P}_\mathcal{V})$ defines a twist on $X \times_\alpha \mathbb{Z}$.
**Theorem.** The Weyl groupoid associated to the Cartan pair $(\mathcal{O}(\Gamma(\mathcal{V}, \alpha)), C(X))$ is isomorphic to $X \times_{\alpha} \mathbb{Z}$.

For every line bundle $\mathcal{V}$ over $X$, one can associate a principal circle bundle $\mathcal{P}_\mathcal{V}$ in a "canonical way".

Define $j_{\alpha}: X \times_{\alpha} \mathbb{Z} \to X$ by $j_{\alpha}(\alpha^k(x), k, x) \mapsto x$, then Deaconu-Kumjian-Muhly showed that $j_{\alpha}^*(\mathcal{P}_\mathcal{V})$ defines a twist on $X \times_{\alpha} \mathbb{Z}$. Let denote this twist by $\Sigma_{\mathcal{V}, \alpha}$. 
**Theorem** Let $\alpha : X \to X$ be a topologically free homeomorphism on compact metric space $X$ and $\mathcal{V} = (V, p, X)$ be a complex line bundle. Then Cartan pairs $(C^r(X \times_{\alpha} \mathbb{Z}, \Sigma_{\mathcal{V}, \alpha}), C(X))$ and $(C(X) \rtimes_{\Gamma(\mathcal{V}, \alpha)} \mathbb{Z}, C(X))$ are isomorphic. In particular, the Weyl twist associated to the Cartan pair $(C(X), C(X) \rtimes_{\Gamma(\mathcal{V}, \alpha)} \mathbb{Z})$ is isomorphic to $\Sigma_{\mathcal{V}, \alpha}$.
**Theorem** Let $\alpha: X \to X$ be a topologically free homeomorphism on compact metric space $X$ and $\mathcal{V} = (V, p, X)$ be a complex line bundle. Then Cartan pairs $(C^*_r(X \times_\alpha \mathbb{Z}, \Sigma_{V,\alpha}), C(X)))$ and $(C(X) \rtimes_{\Gamma(\mathcal{V},\alpha)} \mathbb{Z}, C(X))$ are isomorphic. In particular, the Weyl twist associated to the Cartan pair $(C(X), C(X) \rtimes_{\Gamma(\mathcal{V},\alpha)} \mathbb{Z})$ is isomorphic to $\Sigma_{V,\alpha}$.

So non-trivial line bundles $\Rightarrow$ non-trivial twists.
Thank you for listening!