Subproduct systems with quantum symmetry

(Joint work with Sergey Neshveyev)

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A **subproduct system** is a family $\mathcal{H} = \{H_n\}_{n=0}^\infty$ of finite dimensional Hilbert spaces, $H_0 \cong \mathbb{C}$, together with isometries $w_{k,l} : H_{k+l} \to H_k \otimes H_l$ such that

$$(w_{k,l} \otimes 1)w_{k+l,n} = (1 \otimes w_{l,n})w_{k,l+n} : H_{k+l+n} \to H_k \otimes H_l \otimes H_n.$$ 

We may assume that $\mathcal{H}$ is **standard**:

$$H_n \subset H_1 \otimes n, \quad H_{k+l} \subset H_k \otimes H_l.$$ 

There is a bijection between standard subproduct systems $\mathcal{H}$ with $\dim H_1 = m$ and homogeneous ideals $I \subset \mathbb{C}\langle X_1, X_2, ..., X_m \rangle$ with no polynomials of degree 1:

$$H_n = \{p \in I : \deg(p) = n\} \subset H_1 \otimes n.$$ 

(All of the above: Shalit-Solel 09.)
Associated $C^*$-algebras

We define the Fock space by $\mathcal{F}_{\mathcal{H}} = \bigoplus_n H_n$. For $(e_i)_{i \text{ o.n.b.}}$ in $H_1$

$$S_i : \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{H}}, \quad S_i(\zeta) = w_{1,n}^*(e_i \otimes \zeta), \quad \zeta \in H_n$$

define a bounded linear operators. The associated Toeplitz algebra is

$$\mathcal{T}(\mathcal{H}) = C^*(S_1, S_2, \ldots, S_m) \subset B(\mathcal{F}_{\mathcal{H}}).$$

Observation: $K(\mathcal{F}_{\mathcal{H}}) \subset \mathcal{T}(\mathcal{H})$. The quotient $\mathcal{O}(\mathcal{H}) = \mathcal{T}(\mathcal{H})/K(\mathcal{F}_{\mathcal{H}})$ is called the associated Cuntz-Pimsner algebra (Viselter).

Some studied subproduct systems:

- $d$-contractions: $I = \langle X_1 X_2 - X_2 X_1 \rangle$ (Popescu, Arveson).

- $q$-commuting $d$-contractions: $I = \langle X_1 X_2 - qX_2 X_1 \rangle$.
  (Arias-Popescu, Bhat-Bhattacharyya)

- $G$-subproduct systems (Andersson), $G = SU(2)$ (Arici-Kaad).
Let $G$ be a compact (quantum) group.

A subproduct system $\mathcal{H}$ is called $G$-equivariant, or a $G$-subproduct system if it consists of unitary representations of $G$ and the isometries $H_{k+1} \rightarrow H_k \otimes H_l$ are intertwiners.

Then we get a representation of $G$ on $\mathcal{F}_{\mathcal{H}}$, and the action $G \curvearrowright \mathcal{B}(\mathcal{F}_{\mathcal{H}})$ restricts to $K(\mathcal{F}_{\mathcal{H}})$, $\mathcal{T}(\mathcal{H})$. We get an equivariant exact sequence

$$0 \rightarrow K(\mathcal{F}_{\mathcal{H}}) \rightarrow \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{O}(\mathcal{H}) \rightarrow 0.$$ 

Example (Arveson 98’)

Consider the $SU(2)$-subproduct system $\mathcal{H}$ associated to

$$I = \langle X_1 X_2 - X_2 X_1 \rangle.$$ 

Then $\mathcal{O}(\mathcal{H}) \cong C(Y)$ for some $Y \subset S^3$. The surjective $*$-homomorphism $C(S^3) \rightarrow C(Y)$ is $SU(2)$-equivariant. Hence $Y \cong S^3$ because $SU(2) \curvearrowright S^3$ is transitive.
Temperley-Lieb polynomials

Let $H$ be a Hilbert space, $m = \dim H \geq 2$. Let $F = (a_{ij})_{i,j} \in M_m \mathbb{C}$ and $F^c = (\bar{a}_{ij})_{i,j}$. Consider a quadratic polynomial

$$P = \sum_{i,j=1}^{m} a_{ij} X_i X_j \in H \otimes H.$$ 

$P$ is called Temperley-Lieb if the orthogonal projection $e : H \otimes H \rightarrow \mathbb{C}P$ satisfies

$$(e \otimes 1)(1 \otimes e)(e \otimes 1) = \frac{1}{\lambda}(e \otimes 1), \quad \lambda > 0.$$ 

Lemma

$P$ is Temperley-Lieb if and only if $FF^c$ is unitary up to a scalar.

We may assume

$$P = \sum_{i=1}^{m} a_i X_i X_{m-i+1}, \quad |a_i a_{m-i+1}| = 1, \quad \lambda^{1/2} = \sum_{i=1}^{m} |a_i|^2 \geq m.$$
Some properties of $\mathcal{H}_P$

Let $\mathcal{H}_P = \{H_n\}_n$ be the subproduct system corresponding to a Temperley-Lieb polynomial $P$. By definition $H_n = f_n H \otimes^n$, where

$$f_0 = 1_C, \quad f_1 = 1_H, \quad f_n = 1 - \bigvee_{n=0}^{n-2} 1 \otimes^i e \otimes 1 \otimes^{n-i-2}.$$

The projection $e$ defines Temperley-Lieb algebras:

$$\mathcal{T}\mathcal{L}_n(\lambda^{-1}) \cong C^* (1 \otimes^i e \otimes 1 \otimes^{n-i-2} : 0 \leq i \leq n-2) \subset B(H \otimes^n).$$

So we can use results by Jones 83’:

$$f_{n+1} = 1 \otimes f_n - f_{n-1}', \quad f_{n-1}' \sim e \otimes f_{n-1}.$$

This implies in particular that $\dim H_{n+1} = m \dim H_n - \dim H_{n-1}$ which has the known solution

$$\dim H_n = [n+1]_t = \frac{t^{n+1} - t^{-(n+1)}}{t - t^{-1}}, \quad t + t^{-1} = m.$$
Relations in $\mathcal{T}(\mathcal{H}_P)$

Let $c \subset K(F_{\mathcal{H}_P})$ be the set of convergent sequences and denote by $
abla : c \to c$ the left shift.

**Proposition**

The following relations hold in $\mathcal{T}(\mathcal{H}_P)$: For $f \in c$, $1 \leq i, j \leq m$

\[
fS_i = S_i \nabla(f), \quad \sum_{i=1}^{m} S_i S_i^* = 1 - e_0, \quad \sum_{i=1}^{m} a_i S_i S_{m-i+1} = 0,
\]

\[
S_i^* S_j + a_i \bar{a}_j \phi S_{m-i+1} S_{m-j+1}^* = \delta_{ij},
\]

where $\phi(n) = \frac{[n]_q}{[n+1]_q}$ and $q + q^{-1} = \sum |a_i|^2$.

Note: $\phi(n) \to q$.

The proof uses relations in $\mathcal{T}_L(\lambda^{-1})$ due to Jones and Wenzl.
A special class

Consider a Temperley-Lieb polynomial $P$ with $FF^c = \pm 1$. Then we have a related quantum group:

### Definition (Free orthogonal quantum group, Wang-Van Daele)

$C(O_F^\pm)$ is the universal unital $C^*$-algebra generated by $u_{ij}$, $1 \leq i, j \leq m$ such that $U = (u_{ij})_{i,j}$ is unitary and $U = FU^cF^{-1}$. The coproduct is

$$\Delta : C(O_F^\pm) \rightarrow C(O_F^\pm) \otimes C(O_F^\pm), \quad \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

By definition $(U \otimes U)(P \otimes 1) = (P \otimes 1)$. Hence

- $\mathcal{H}_P = \{H_n\}_n$ is $O_F^\pm$-equivariant,
- $\{H_n\}_n$ consists of representatives for the isomorphism classes of irreducible representations of $O_F^\pm$ (Banica).

In Arici-Kaad (20’) $SU(2)$-subproduct systems are considered. These correspond to $P = \sum_{i=1}^m (-1)^i X_i X_{m-i+1}$ and an inclusion $SU(2) \subset O_F^\pm$. 

$$P = \sum_{i=1}^m a_i X_i X_{m-i+1}, \quad FF^c = \pm 1$$
The 2-dimensional case

When \( m = 2 \) and \( FF^c = \pm 1 \) we have \( O^+_F = SU_q(2) \) for \( q \in \mathbb{R}^\times \).

**Definition (Woronowicz)**

For \( q \in \mathbb{R}^\times \), \( C(SU_q(2)) \) is the universal \( C^* \)-algebra generated by \( \alpha, \gamma \) subject to the relations

\[
\alpha^* \alpha + \gamma^* \gamma = 1, \quad \alpha \alpha^* + q^2 \gamma \gamma^*,
\]

\[
\gamma^* \gamma = \gamma \gamma^*, \quad \alpha \gamma = q \gamma \alpha, \quad \alpha \gamma^* = q \gamma^* \alpha.
\]

**Proposition \( (q \in \mathbb{R}^\times) \)**

There is a \( SU_q(2) \)-equivariant \(*\)-isomorphism \( C(SU_q(2)) \to \mathcal{O}(\mathcal{H}_P) \).

For any \( q \in \mathbb{C}^\times \) we can define \( U_q(2) \) (Zhang-Zhao). There is a \( U_q(2) \)-equivariant \(*\)-isomorphism \( C(U_q(2)/T) \to \mathcal{O}(\mathcal{H}_P) \).
Monoidal equivalence

Let $G_1$, $G_2$ be compact quantum groups. They are called monoidally equivalent if their representation categories are ($C^*$-tensor) equivalent.

A linking algebra for $G_1$ and $G_2$ is $C^*$-algebra $B(G_1, G_2)$ with commuting free ergodic actions by $G_1$ and $G_2$.

Theorem (Ulbrich, Bichon-DeRijdt-Vaes)

$G_1$ and $G_2$ are monoidally equivalent if and only if there is a linking algebra $B(G_1, G_2)$. The equivalence $\text{Rep } G_1 \rightarrow \text{Rep } G_2$ is given by

$$H_U \mapsto H_U \boxtimes_{G_1} B(G_1, G_2).$$

De-Rijdt-Vander Vennet: The functor $\cdot \boxtimes_{G_1} B(G_1, G_2)$ defines an equivalence between the categories of $G_i$-$C^*$-algebras.

For $F \in \text{GL}_m(\mathbb{C})$ with $FF^c = \pm 1$ there is a monoidal equivalence between $O_F^+$ and $SU_q(2)$ for some $q \in \mathbb{R}^\times$ (Bichon-DeRijdt-Vaes).
Theorem

Assume $G_1$ and $G_2$ are monoidally equivalent compact quantum groups, and that $\mathcal{H}_1 = \{H_n\}_n$ is a $G_1$-subproduct system. Then

- $\mathcal{H}_2 = \{H_n \boxtimes_{G_1} B(G_1, G_2)\}_n$ is a $G_2$-subproduct system,
- $T(\mathcal{H}_2) \cong T(\mathcal{H}_1) \boxtimes_{G_1} B(G_1, G_2)$
- $O(\mathcal{H}_2)_r \cong O(\mathcal{H}_1)_r \boxtimes_{G_1} B(G_1, G_2)$

Corollary

For $P = \sum_i a_i X_i X_{m-i+1}$ with $FF^c = \pm 1$ there is $q \in \mathbb{R}^\times$ such that

$$O(\mathcal{H}_P) \cong C(SU_q(2) \boxtimes SU_q(2)) B(SU_q(2), O_F^+) \cong B(SU_q(2), O_F^+)$$

Using this we can show that $T(\mathcal{H}_P)$ is the universal $C^*$-algebra with the relations given earlier, and also compute $KK$-theory.
Let $P \in H \otimes H$ be a Temperley-Lieb polynomial.

**Proposition**

There is a compact (matrix) quantum group $G$ with a representation $V \in B(H) \otimes C(G)$ and a grouplike element $x \in C(G)$ such that

$$(V \otimes V)(P \otimes 1) = P \otimes x.$$ 

The condition above says that the projection $e : H \otimes H \to \mathbb{C}P$ is an intertwiner from $V \otimes V$ to the one-dimensional representation $1 \otimes x$. It follows that $\mathcal{H}_P$ is $G$-equivariant.

- $G = " T \ltimes O_F^+ "$ where $T$ is a compact abelian group, and $O_F^+$ is a braided quantum group. $x$ comes from $T$.
- Special cases of braided $O_F^+$ have been studied (Kasprzak, Meyer, Roy, Woronowicz).
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