On the Lifting of the Dirac Elements in the Higson-Kasparov Theorem

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Plan of the Talk

Main Topic: The Higson-Kasparov Theorem (2001)
$=$ The Baum-Connes Conjecture for a-T-menable groups

- What is the main idea for proving this theorem
- What kind of interesting analysis comes into play (Non-commutative functional calculus: defenitions and corrections)
- Simplifications in the lifting argument from E-theory to KK-theory

a-T-menable Groups

Definition: a-T-menable groups

A second countable locally compact group is a-T-menable if it acts metrically properly and affine isometrically on a (real) Hilbert space.

Examples of a-T-menable groups

- All compact groups.
- All amenable groups (e.g. abelian groups).
- Groups which act properly on trees (e.g. free groups F_n).
- All closed subgroups of $SO(n,1)$ or $SU(n,1)$.

Non-examples

- $Sp(n,1)$ for $n \geq 2$, $SL(n,\mathbb{Z})$, $SL(n,\mathbb{R})$ for $n \geq 3$ (Property (T));
- $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ (relative Property (T)).

BCC

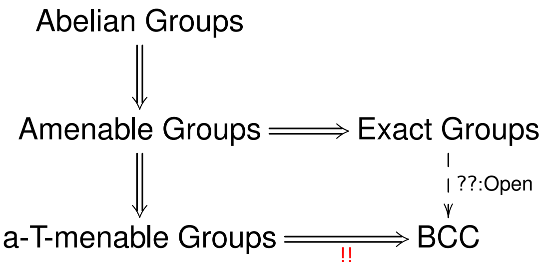
- G: a second countable, locally compact topological group.
- $\mathcal{E}G$: a classifying space for proper actions of G.
- A: a separable G-C^*-algebra.
- $A \rtimes_r G$: the reduced crossed product of A.

The Baum-Connes Conjecture with Coefficients (BCC), 1991

$$RKK_*^G(\mathcal{E}G, A) \cong K_*(A \rtimes_r G) \quad (* = 0, 1)$$

- LHS = Equivariant K-homology of $\mathcal{E}G$ with coefficient in A.
- RHS = K-theory of $A \rtimes_r G$.
- G satisfies BCC $\Rightarrow$ All closed subgroups of G satisfy BCC.
- BCC $\Rightarrow$ The Baum-Connes Conjecture (when $A = \mathbb{C}$).

BCC and Higson-Kasparov Theorem



!!: The Higson-Kasparov Theorem (2001)

Any second countable a-T-menable group *G* satisfies *BCC*

Their proof involves (infinite-dimensional) functional analysis; (no classical differential geometry or representation theory).

Setting

- $\mathcal{H}$: a separable (real) Hilbert space.
- G: an a-T-menable group acting properly on $\mathcal{H}$.

Dual Dirac Method ($1_G = \gamma_G$)

- KK^G: Kasparov's equivariant KK-theory

The Dual Dirac Method

The following is sufficient to prove BCC for G:

- Find a proper G-C^*-algebra $A(\mathcal{H})$;
- Find an element $b \in KK_*^G(\mathbb{C}, A(\mathcal{H}))$ (Bott element);
- Find an element $d \in KK_*^G(A(\mathcal{H}), \mathbb{C})$ (Dirac element);

such that

- The product $b \otimes_{A(\mathcal{H})} d = 1_G$ in $KK^G(\mathbb{C}, \mathbb{C})$.

A G-C^*-algebra A is proper if there is a proper G-space X and a nondegenerate equivariant $*$-homomorphism from $C_0(X)$ to the center of the multiplier algebra $M(A)$.

Strategy

There is a finite dimensional version of what we want to do:

When Hilbert space is finite dimensional ($\mathcal{H} = \mathbb{R}^n$)

- $C_\tau(\mathbb{R}^n) = \Gamma_0(\mathbb{R}^n, \mathrm{Cliff}(\mathbb{R}^n))$ (proper algebra);
- $b_{\mathbb{R}^n} \in KK_0^G(\mathbb{C}, C_\tau(\mathbb{R}^n))$ (Bott element);
- $d_{\mathbb{R}^n} \in KK_0^G(C_\tau(\mathbb{R}^n), \mathbb{C})$ (Dirac element);
- $b_{\mathbb{R}^n} \otimes_{C_\tau(\mathbb{R}^n)} d_{\mathbb{R}^n} = [L^2(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R}^n)),\ B_{\mathbb{R}^n}]$;
- $[L^2(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R}^n)),\ B_{\mathbb{R}^n}] = 1_G$ in $KK^G(\mathbb{C}, \mathbb{C})$.

Here, $B_{\mathbb{R}^n}$ is a Bott-Dirac operator of $\mathbb{R}^n$.

n=1

- $c(e_1)$ ($\bar{c}(e_1)$): (skew-) s.a. Clifford mult. by $e_1 \in \mathbb{R}$;
- $b_{\mathbb{R}} = [C_\tau(\mathbb{R}), c(e_1)x] \in KK_0^G(\mathbb{C}, C_\tau(\mathbb{R}))$;
- $d_{\mathbb{R}} = [C_\tau(\mathbb{R}) \curvearrowright L^2(\mathbb{R}, \Lambda_*^{\mathbb{C}}(\mathbb{R})), \bar{c}(e_1)\frac{d}{dx}] \in KK_0^G(C_\tau(\mathbb{R}), \mathbb{C})$;
- $b_{\mathbb{R}} \otimes_{C_\tau(\mathbb{R})} d_{\mathbb{R}} = [L^2(\mathbb{R}, \Lambda_*^{\mathbb{C}}(\mathbb{R})), B_{\mathbb{R}} = c(e_1)x + \bar{c}(e_1)\frac{d}{dx}]$.

Strategy

The proof of the Higson-Kasparov Theorem is nothing but to make sense of the following “limits”:

- $A(\mathcal{H}) :=$ “$\lim C_\tau(\mathbb{R}^n)$” (proper algebra);
- $b :=$ “$\lim b_{\mathbb{R}^n}$” $\in KK_1^G(\mathbb{C}, A(\mathcal{H}))$ (Bott element);
- $d :=$ “$\lim d_{\mathbb{R}^n}$” $\in KK_1^G(A(\mathcal{H}), \mathbb{C})$ (Dirac element);
- “$\lim[L^2(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R}^n)),\ B_{\mathbb{R}^n}]$” $= 1_G \in KK^G(\mathbb{C}, \mathbb{C})$.

An interesting analysis is used when we deal with the “limit” of the cycles $[L^2(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R}^n)),\ B_{\mathbb{R}^n}]$.
(Non-commutative functional calculus).

Bott-Dirac Operator of $\mathbb{R}^n$

The Bott-Dirac Operator $B_{\mathbb{R}^n}$ represents 1_G in $KK^G(\mathbb{C},\mathbb{C})$.

$$B_{\mathbb{R}} := c(e_1)x + \bar{c}(e_1)\frac{d}{dx} = \begin{pmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{pmatrix}$$

- $B_{\mathbb{R}}$ is an odd unbounded operator on $L^2(\mathbb{R}, \Lambda^{\mathbb{C}}_*(\mathbb{R}))$;
- It is defined on the Schwartz space $s(\mathbb{R}, \Lambda^{\mathbb{C}}_*(\mathbb{R}))$;
- It is selfadjoint and diagonalizable;
- It has compact resolvent; and $\mathrm{Ker}B_{\mathbb{R}} = \mathrm{span}\{e^{-\frac{\|x\|^2}{2}}\}$.

$$B_{\mathbb{R}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix} & 0 & 0 & \cdots \\ 0 & 0 & \begin{pmatrix} 0 & \sqrt{4} \\ \sqrt{4} & 0 \end{pmatrix} & 0 & \cdots \\ 0 & 0 & 0 & \begin{pmatrix} 0 & \sqrt{6} \\ \sqrt{6} & 0 \end{pmatrix} & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Bott-Dirac Operator of $\mathbb{R}^n$

$$B_{\mathbb{R}^n} := \sum_{j=1}^{n} c(e_j)x_j + \bar{c}(e_j)\frac{\partial}{\partial x_j}$$

$$= B_{\mathbb{R}}\hat{\otimes}1\hat{\otimes}\cdots\hat{\otimes}1 + 1\hat{\otimes}B_{\mathbb{R}}\hat{\otimes}1\hat{\otimes}\cdots\hat{\otimes}1 + \cdots + 1\hat{\otimes}\cdots\hat{\otimes}1\hat{\otimes}B_{\mathbb{R}}$$

$$L^2(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R}^n)) = L^2(\mathbb{R}, \Lambda_*^{\mathbb{C}}(\mathbb{R}))\hat{\otimes}\cdots\hat{\otimes}L^2(\mathbb{R}, \Lambda_*^{\mathbb{C}}(\mathbb{R}))$$

- $B_{\mathbb{R}^n}$ is an odd unbounded operator on $L^2(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R}^n))$;
- It is defined on the Schwartz space $s(\mathbb{R}^n, \Lambda_*^{\mathbb{C}}(\mathbb{R})^n)$;
- It is independent of the choice of a basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$;
- It is selfadjoint and diagonalizable;
- It has compact resolvent; and $\mathrm{Ker} B_{\mathbb{R}^n} = \mathrm{span}\{e^{-\frac{\|x\|^2}{2}}\}$;
- Note: the eigenspace of $B_{\mathbb{R}^n}^2$ for $\lambda = 2$ has dimension $2n$.

Bott-Dirac Operator of $\mathcal{H}$

We want to define Bott-Dirac Operator $B_{\mathcal{H}}$ of an infinite dimensional Hilbert space $\mathcal{H}$ as an inductive limit of $B_{\mathbb{R}^n}$.

There is a natural construction of such an inductive limit.

Bott-Dirac Operator of $\mathcal{H}$

- $\mathcal{H}$: a separable (real) (G-)Hilbert space.
- For each finite dimensional subspace V of $\mathcal{H}$,

  $H(V) := L^2(V, \Lambda_*^{\mathbb{C}}(V))$;

  $s(V) := s(V, \Lambda_*^{\mathbb{C}}(V))$ (Schwartz space);

  B_V: Bott-Dirac Operator of V
- For an inclusion of subspaces $V \subset V' = V \oplus W$,

  $H(V) \to H(V') = H(V)\hat{\otimes}H(W)$: $\xi \mapsto \xi\hat{\otimes}e^{-\frac{\|w\|^2}{2}}$
- $H(\mathcal{H}) := \lim_{V} H(V)$: naturally G-Hilbert space.

$$B_{\mathcal{H}} := \lim_{V} B_V$$

- $B_{\mathcal{H}}$ is defined on $s(\mathcal{H}) := \text{alg-}\lim_{V} s(V)$;
- It is a well-defined odd unbounded operator on $H(\mathcal{H})$.

$$B_{\mathcal{H}} := \lim_{V} B_V$$

For $\xi \in s(V) \subset s(\mathcal{H})$, $B_{\mathcal{H}}(\xi) := B_V \xi \in s(V) \subset s(\mathcal{H})$.

It is well-defined because
the following diagram commutes for $V \subset V \oplus W \subset \mathcal{H}$:

$$\begin{array}{ccc} s(V) & \longrightarrow & s(V \oplus W) \\ \downarrow{\scriptstyle B_V} & & \downarrow{\scriptstyle B_{V\oplus W}} \\ s(V) & \longrightarrow & s(V \oplus W) \end{array}$$

To see this, one may write $B_{V\oplus W} = B_V + B_W$. For $\xi \in s(V)$,

$$B_{V\oplus W}(\xi \hat{\otimes} e^{-\frac{\|w\|^2}{2}}) = B_V \xi \hat{\otimes} e^{-\frac{\|w\|^2}{2}} + \xi \hat{\otimes} B_W e^{-\frac{\|w\|^2}{2}} = B_V \xi \hat{\otimes} e^{-\frac{\|w\|^2}{2}}.$$

Bott-Dirac Operator of $\mathcal{H}$

$$B_{\mathcal{H}} := \lim_{V} B_V$$

- $B_{\mathcal{H}}$ is an odd unbounded operator on $H(\mathcal{H}) := \lim_{V} H(V)$;
- It is selfadjoint and diagonalizable;
- It is G-equivariant if G acts on $\mathcal{H}$ linearly;
- $\mathrm{Ker} B_{\mathcal{H}} = \mathrm{span}\{e^{-\frac{\|x\|^2}{2}}\}$;
- It does not have compact resolvent:

  the eigenspace of $B_{\mathcal{H}}^2$ for $\lambda = 2$ has infinite dimension.

$[H(\mathcal{H}), B_{\mathcal{H}}]$ doesn't define an element in $KK^G(\mathbb{C}, \mathbb{C})$.

Quick Solution

After fixing some basis $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}$, we may write $B_{\mathcal{H}}$ as:

$$B_{\mathcal{H}} = \sum_{j=1}^{\infty} c(e_j)x_j + \bar{c}(e_j)\frac{\partial}{\partial x_j}$$

$$= B_{\mathbb{R}}\hat{\otimes}1\hat{\otimes}\cdots + 1\hat{\otimes}B_{\mathbb{R}}\hat{\otimes}1\hat{\otimes}\cdots + \cdots$$

$$H(\mathcal{H}) = L^2(\mathbb{R}, \Lambda_*^{\mathbb{C}}(\mathbb{R}))\hat{\otimes}L^2(\mathbb{R}, \Lambda_*^{\mathbb{C}}(\mathbb{R}))\hat{\otimes}\cdots$$

A quick solution for the non-compact resolvent issue is:

Proposition

Fix some basis as above. For any unbounded increasing sequence (n_k) of positive numbers, $\tilde{B}_{\mathcal{H}} := n_1 B_{\mathbb{R}}\hat{\otimes}1\hat{\otimes}\cdots + n_2 1\hat{\otimes}B_{\mathbb{R}}\hat{\otimes}1\hat{\otimes}\cdots + n_3 1\hat{\otimes}1\hat{\otimes}B_{\mathbb{R}}\hat{\otimes}\cdots + \cdots$ defines an unbounded, diagonalizable selfadjoint operator on $H(\mathcal{H})$ having compact resolvent with $\mathrm{Ker}B_{\mathcal{H}} = \mathrm{span}\{e^{-\frac{\|x\|^2}{2}}\}$.

Non-Commutative Functional Calculus

Non-commutative functional calculus is more systematic way to do such perturbation of $B_{\mathcal{H}}$.

Non-Commutative Functional Calculus (Higson, Kasparov)

- h: any symmetric, densely defined operator on $\mathcal{H}$;
- $\mathcal{H}_h$: the domain of h;
- $h(B_{\mathcal{H}}) := \sum_{j=1}^{\infty} c(he_j)x_j + \bar{c}(he_j)\frac{\partial}{\partial x_j}$;
- $h(B_{\mathcal{H}})$ is defined on alg-$\lim_{V \subset H_h} s(V)$;
- It is a well-defined odd symmetric operator on $H(\mathcal{H})$;
- It is independent of the choice of a basis $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}_h$;
- When h is diagonalizable, so is $h(B_{\mathcal{H}})$;
- When h has compact resolvent, so is $h(B_{\mathcal{H}})$;
- The assignment $h \mapsto h(B_{\mathcal{H}})$ is $\mathbb{R}$-linear.

Non-commutative Functional Calculus

Let's take a closer look at the definition:

Consider for any densely defined operator h on $\mathcal{H}$,

- $V_n := \text{span}\{\, e_j \mid j = 1, \cdots, n \,\} \subset \mathcal{H}_h$
- $h(B_{V_n}) := \displaystyle\sum_{j=1}^{n} c(he_j)x_j + \bar{c}(he_j)\frac{\partial}{\partial x_j}$

$$\text{Can we define } h(B_{\mathcal{H}}) := \lim h(B_{V_n})?;$$

For $V_n \subset V_{n'} \subset V_{n''} \subset \mathcal{H}_h$ and $V_{n''} + hV_{n''} \subset W$:

we may hope that the following diagram commutes:

$$\begin{array}{ccccc}
s(V_n, \Lambda_*^{\mathbb{C}}(V_n)) & \longrightarrow & s(V_{n'}, \Lambda_*^{\mathbb{C}}(V_{n'})) & \longrightarrow & s(V_{n''}, \Lambda_*^{\mathbb{C}}(V_{n''})) \\
 & & \big\downarrow{\scriptstyle h(B_{V_{n'}})} & & \big\downarrow{\scriptstyle h(B_{V_{n''}})} \\
 & & s(V_{n'}, \Lambda_*^{\mathbb{C}}(W)) & \longrightarrow & s(V_{n''}, \Lambda_*^{\mathbb{C}}(W))
\end{array}$$

Non-commutative Functional Calculus

In the paper by Higson and Kasparov,
it was (implicitly) claimed that this diagram commutes:

$$\begin{array}{ccccc} s(V_n, \Lambda^{\mathbb{C}}_*(V_n)) & \longrightarrow & s(V_{n'}, \Lambda^{\mathbb{C}}_*(V_{n'})) & \longrightarrow & s(V_{n''}, \Lambda^{\mathbb{C}}_*(V_{n''})) \\ & & \downarrow h(B_{V_{n'}}) & & \downarrow h(B_{V_{n''}}) \\ & & s(V_{n'}, \Lambda^{\mathbb{C}}_*(W)) & \longrightarrow & s(V_{n''}, \Lambda^{\mathbb{C}}_*(W)) \end{array}$$

However, in general, this diagram does not commute.

Theorem (Fixed non-commutative functional calculus) (N.)

- The diagram asymptotically commutes iff h^* is defined on V_n.
- The following formula defines $h(B_{\mathcal{H}})$ unambiguously for any h whose adjoint h^* is defined on $\mathcal{H}_h$:

$$\xi \in s(V),\ h(B_{\mathcal{H}})(\xi) := \lim_{\substack{W \subset \mathcal{H}_h \\ W \perp V}} h(B_{V \oplus W})(\xi \otimes e^{-\frac{\|w\|^2}{2}})$$

Non-commutative Functional Calculus

Example 1: consider if $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on span$\{e_1, e_2\}$;

$$\begin{array}{ccccc} s(V_1, \Lambda_*^{\mathbb{C}}(V_1)) & \longrightarrow & s(V_1, \Lambda_*^{\mathbb{C}}(V_1)) & \longrightarrow & s(V_2, \Lambda_*^{\mathbb{C}}(V_2)) \\ & & \Big\downarrow h(B_{V_1}) & & \Big\downarrow h(B_{V_2}) \\ & & s(V_1, \Lambda_*^{\mathbb{C}}(W)) & \longrightarrow & s(V_2, \Lambda_*^{\mathbb{C}}(W)) \end{array}$$

This is not commutative due to the following nonzero term:

$$(c(he_2)x_2 + \bar{c}(he_2)\tfrac{\partial}{\partial x_2})(\xi \hat{\otimes} e^{-\frac{x_2^2}{2}})$$

$$= \mathrm{int}(e_1)\xi \hat{\otimes} 2x_2 e^{-\frac{x_2^2}{2}} \text{ (for } \xi \in s(V_1)).$$

Non-commutative Functional Calculus

Example 2: consider:

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \end{pmatrix} (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots);$$

$$\begin{array}{ccccc} s(V_1, \Lambda_*^{\mathbb{C}}(V_1)) & \longrightarrow & s(V_{n'}, \Lambda_*^{\mathbb{C}}(V_{n'})) & \longrightarrow & s(V_{n''}, \Lambda_*^{\mathbb{C}}(V_{n''})) \\ & & \downarrow h(B_{V_{n'}}) & & \downarrow h(B_{V_{n''}}) \\ & & s(V_{n'}, \Lambda_*^{\mathbb{C}}(W)) & \longrightarrow & s(V_{n''}, \Lambda_*^{\mathbb{C}}(W)) \end{array}$$

This is not commutative no matter how large we take $V_{n'}$. Nonetheless, it asymptotically commutes.

Non-commutative Functional Calculus

- So what is "$\lim[L^2(\mathbb{R}^n, \Lambda^{\mathbb{C}}_*(\mathbb{R}^n)),\ B_{\mathbb{R}^n}]$" ?
- With suitable h, one can guarantee "asymptotic equivariance" for a family $\{(1+th)(B_{\mathcal{H}})\}_{t>0}$.
- "Via asymptotic morphisms", one may define "$[H(\mathcal{H}), (1+th)(B_{\mathcal{H}})]$"$= 1_G \in KK^G(\mathbb{C}, \mathbb{C})$.

$A(\mathcal{H})$: C^*-algebra of Hilbert space $\mathcal{H}$

The following "limits" can be defined without so much trouble:

- $A(\mathcal{H}) :=$ "$\lim C_\tau(\mathbb{R}^n)$" (proper algebra);
- $b :=$ "$\lim b_{\mathbb{R}^n}$" (Bott element);

- $S = C_0(\mathbb{R})$: a (graded) C^*-algebra.

$A(\mathcal{H}) := \lim_V S\hat{\otimes}C_\tau(V)$ is defined as follows:

- For an inclusion of subspaces $V \subset V' = V \oplus W$,

  $S\hat{\otimes}C_\tau(V) \to S\hat{\otimes}C_\tau(V') \cong S\hat{\otimes}C_\tau(W)\hat{\otimes}C_\tau(V)$:

It's given by the graded tensor product of $*$-homomorphisms:

- $S \to S\hat{\otimes}C_\tau(W) : f \mapsto f(x\hat{\otimes}1 + 1\hat{\otimes}c_W)$

- $C_\tau(V) \to C_\tau(V)$: the identity on $C_\tau(V)$

$A(\mathcal{H})$: C^*-algebra of Hilbert space $\mathcal{H}$

Explanation for $S \to S\hat{\otimes}C_\tau(W) : f \mapsto f(x\hat{\otimes}1 + 1\hat{\otimes}c_W)$:

- x is an (odd) unbounded multiplier on S: multiplication by x at x in $\mathbb{R}$;
- c_W is an (odd) unbounded multiplier on $C_\tau(W)$: Clifford multiplication $c(w)$ at w in W;

$x\hat{\otimes}1 + 1\hat{\otimes}c_W$ is an (odd) unbounded multiplier on $S\hat{\otimes}C_\tau(W)$.

We have a functional calculus $f \mapsto f(x\hat{\otimes}1 + 1\hat{\otimes}c_W)$.

For example:

$e^{-x^2} \mapsto e^{-x^2}\hat{\otimes}e^{-\|w\|^2}$;

$xe^{-x^2} \mapsto xe^{-x^2}\hat{\otimes}e^{-\|w\|^2} + e^{-x^2}\hat{\otimes}c_W e^{-\|w\|^2}$.

$A(\mathcal{H})$: C^*-algebra of Hilbert space $\mathcal{H}$

- When a group G acts on $\mathcal{H}$ affine isometrically,

  the C^*-algebra $A(\mathcal{H})$ is naturally a G-C^*-algebra.

- If moreover, G acts on $\mathcal{H}$ (metrically) properly,

  the C^*-algebra $A(\mathcal{H})$ is a proper G-C^*-algebra.

- Indeed, the center of $A(\mathcal{H})$ is $C_0([0,\infty)\times\mathcal{H})$

- $\lim_W x\hat{\otimes}1 + 1\hat{\otimes}c_W$ defines an element b in $KK_1^G(\mathbb{C}, A(\mathcal{H}))$.

“$\lim d_{\mathbb{R}^n}$” and Spectral Dual-Dirac

- It is somewhat technical to construct the Dirac element $d :=$ “$\lim d_{\mathbb{R}^n}$” with bare hands in KK^G.

What we can simply have is the following “Spectral Dual Dirac”

- $A \to\to B$ denotes an asymptotic morphism from A to B
- $A(\mathcal{H})$: the C*algebra of Hilbert space $\mathcal{H}$ (proper algebra)
- $\beta : S \to\to A(\mathcal{H})$ (“Bott element”);
- $\alpha : A(\mathcal{H}) \to\to S\hat{\otimes}K(H(\mathcal{H}))$ (“Dirac element”);
- The composition $\alpha \circ \beta : S \to\to S\hat{\otimes}K(H(\mathcal{H}))$

  is homotopic to $\mathrm{id}_S : S \to\to S$ in a suitable sense.

Indeed, the idea of Higson and Kasparov was to translate (lift) everything into the language of KK-theory:

Proof of Higson-Kasparov Theorem

Higson and Kasparov lifted this Spectral Dual-Dirac to KK in the following way:

- We already have a Bott element $b \in KK_1^G(\mathbb{C}, A(\mathcal{H}))$ which "corresponds to" the "Bott element" $\beta : S \to\to A(\mathcal{H})$;
- We can construct an extension of G-C^*-algebras

$$0 \longrightarrow J \longrightarrow B \longrightarrow A(\mathcal{H}) \longrightarrow 0$$

  which "corresponds to" the "Dirac element"
  $\alpha : A(\mathcal{H}) \to\to S\hat{\otimes}K(H(\mathcal{H}))$;
- Although this extension may not be equivariant semi-split, we can show there is an element d in $KK_1^G(A(\mathcal{H}), J)$ which "corresponds to" this extension;
- One can compute the Kasparov product $b \otimes_{A(\mathcal{H})} d \cong 1_G$.

Spectral Dual-Dirac lifts

The following simplifies what actually happened in the proof:

Theorem (Spectral Dual-Dirac lifts) (N.)

For any G, suppose we have the following "Spectral Dual-Dirac":

- H: a (complex, graded) G-Hilbert space;
- A: a proper nuclear C^*-algebra;
- $\beta : S \to\to A$ ("Bott element");
- $\alpha : A \to\to S\hat{\otimes}K(H)$ ("Dirac element");
- The composition $\alpha \circ \beta : S \to\to S\hat{\otimes}K(H)$

  is "homotopic" to $\mathrm{id}_S : S \to\to S$.

Then, this lifts to Dual-Dirac ($\gamma_G = 1_G$) in KK-theory
if there is a $b \in KK_1^G(\mathbb{C}, A)$ which "corresponds to" β.
i.e. The Spectral Dual-Dirac lifts if the Bott element lifts.

Note, in our setting,
$A(\mathcal{H})$ is nuclear and the Bott element indeed lifts.
This concludes the proof of the Higson-Kasparov Theorem.

Summary

- The proof of the Higson-Kasparov Theorem is nothing but precisely making sense of a limit of Dual-Dirac method for finite dimensional case.
- A noncommutative functional calculus is needed since the naive Bott-Dirac operator in infinite dimensions does not have compact resolvent. Fixed version of this not only gives a precise formula which was not mentioned in the work by Higson and Kasparov but also shows one can apply it with respect to any bounded operators.
- The lifting of the Dual-Dirac method from E-theory to KK-theory can be simplified.

THANK YOU VERY MUCH!!

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