# Shilov boundary for "holomorphic functions" on a quantum matrix ball

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# Definition of Shilov Boundary

The classical notion of a Shilov boundary

- Was introduced by Georgii Shilov.
- It generalizes the maximus modulus priciple:
- If $f(z) \in C(\overline{\mathbb{D}})$ is an holomorphic function on the unit disc $\mathbb{D}$, then

$$\max_{z\in\overline{\mathbb{D}}} |f(z)| = \max_{z\in\mathbb{T}} |f(z)|,$$

where $\mathbb{T}$ is the unit circle.

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# Definition of Shilov Boundary

If $X$ is a compact space and let $\mathcal{A} \subseteq C(X)$ be a uniform algebra.

## Definition

A *boundary* of $X$ relative to $\mathcal{A}$ is a closed subset $B \subseteq X$, such that for every $f(z) \in \mathcal{A}$

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A boundary $S$ is called the *Shilov boundary* of $X$ relative to $\mathcal{A}$ if it is contained in any other boundary.

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In $C(\overline{\mathbb{D}})$ let $\mathcal{A}(\overline{\mathbb{D}}) \subseteq C(\overline{\mathbb{D}})$ be the holomorphic functions. The Shilov boundary of $\overline{\mathbb{D}}$ relative to $\mathcal{A}(\overline{\mathbb{D}})$ is then $\mathbb{T}$.

## Example

Let $\overline{\mathbb{D}}_n$ be the closed unit ball in $M_n(\mathbb{C})$, then the sub-algebra $\mathcal{A}(\overline{\mathbb{D}}_n) \subseteq C(\overline{\mathbb{D}}_n)$ of holomorphic functions has as its Shilov boundary $U_n$, the Lie group of unitary $n \times n$ matrices.

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# Definition of Shilov Boundary

A non-commutative version of the Shilov boundary was proposed by Arveson ['70, Acta Math.]:

$C(X) \rightsquigarrow$ unital $C^*$-algebras $\mathcal{B}$ (noncom. topological spaces)

$K \subset X$, closed $\rightsquigarrow$ closed ideals in $\mathcal{B}$

uniform algebra $\rightsquigarrow$ subalgebra $\mathcal{A}$ that generates the $C^*$-algebra $\mathcal{B}$, $1 \in \mathcal{A}$.

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# Definition of Shilov Boundary

Let $D$ be a $C^*$-algebra and $\mathcal{A} \subseteq D$ a unital subalgebra that generates $D$ as a $C^*$-algebra.

**Definition**

An ideal $J$ in $D$ is called a boundary ideal of $D$ relative $\mathcal{A}$ if the canonical map $j : D \to D/J$ is a complete isometry when restricted to $\mathcal{A}$.

A boundary ideal is called a *Shilov boundary ideal* if it contains any other boundary ideal.

Shilov boundary ideal exists and unique (Arveson '70, Hamana '79).

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# The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$

Let $q$ be a constant $0 < q < 1$.

## Definition

$\mathbb{C}[\mathrm{Mat}_n]_q$ is the algebra over $\mathbb{C}$ with generators $\{z_k^j | 1 \leq k, j \leq n\}$ subject to the relations

$$z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha = 0, \qquad a = b \,\&\, \alpha < \beta, \text{ or } a < b \,\&\, \alpha = \beta,$$
$$z_a^\alpha z_b^\beta - z_b^\beta z_a^\alpha = 0, \qquad \alpha < \beta \quad \& \quad a > b,$$
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# The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$

$\mathrm{Pol}(\mathrm{Mat}_n)_q$ is a $q$-analog of the polynomial algebra on $\mathrm{Mat}_n$. Its generators are $z_k^j$, $(z_k^j)^*$, $1 \leq k, j \leq n$, and the list of relations is the same as those of $\mathbb{C}[\mathrm{Mat}_n]_q$ but also that

$$(z_b^\beta)^* z_a^\alpha = q^2 \cdot \sum_{a',b'=1}^{n} \sum_{\alpha',\beta'=1}^{n} R_{ba}^{b'a'} R_{\beta\alpha}^{\beta'\alpha'} \cdot z_{a'}^{\alpha'} (z_{b'}^{\beta'})^* + (1-q^2)\delta_{ab}\delta^{\alpha\beta},$$

with $\delta_{ab}$, $\delta^{\alpha\beta}$ being the Kronecker symbols, and coefficients $R_{ij}^{kl}$.

We have $\mathrm{Pol}(\mathrm{Mat}_1)_q \cong \mathrm{Pol}(\mathbb{C})_q$ the "quantum unit disc".

$\mathrm{Pol}(\mathrm{Mat}_n)_q$ was introduced by Vaksman & Co-authors in the 90's.

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# The algebra $\mathbb{C}[GL_n]_q$

The $q$-determinant is given by the formula for $\mathbf{z} = (z_i^j)$

$$\det_q \mathbf{z} = \sum_{s \in S_n} (-q)^{l(s)} z_1^{s(1)} z_2^{s(2)} \dots z_n^{s(n)}$$

with $l(s) = \mathrm{card}\{(i,j) | \; i < j \quad \& \quad s(i) > s(j)\}$.

It is known that $\det_q \mathbf{z}$ is in the center of $\mathbb{C}[\mathrm{Mat}_n]_q$.

## Definition

Let $\mathbb{C}[GL_n]_q$ be localization of $\mathbb{C}[\mathrm{Mat}_n]_q$ w.r.t the multiplicative system $(\det_q \mathbf{z})^{\mathbb{N}}$.

The algebra $\mathbb{C}[GL_n]_q$ is a q-analogue of functions on the Lie group of invertible $n \times n$ matrices.

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# $\mathbb{C}[U_n]_q$ and $\mathbb{C}[SU_n]_q$

Recall that $\mathbb{C}[U_n]_q$ is $\mathbb{C}[GL_n]_q$ with the involution $*$ given by

$$(z^j_k)^* = (-q)^{k+j-2n}(\det_q \mathbf{z})^{-1} \det_q \mathbf{z}^j_k$$

Where $\mathbf{z}^j_k$ is $\mathbf{z}$ with the $k'$th row and $j'$th column deleted.

$\mathbb{C}[SU_n]_q$ is the algebra $\mathbb{C}[U_n]_q/\langle(q^{-n(n-1)/2} - \det_q \mathbf{z})\rangle$.

Letting the generators of $\mathbb{C}[SU_n]_q$ be $t_{kj} := q^{n-k} z^j_k$ then this is a quantum group with co-product $\Delta$, co-unit $\epsilon$ and antipode $S$

$$\Delta(t_{kj}) = \sum_m t_{km} \otimes t_{mj} \quad \epsilon(t_{kj}) = \delta_{kj} \quad S(t_{kj}) = t^*_{jk}$$

# $\mathbb{C}[U_n]_q$ and $\mathbb{C}[SU_n]_q$

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# The co-action of $\mathbb{C}[SU_n]_q$ on $\mathrm{Pol}(\mathrm{Mat}_n)_q$

Then there is a co-action of $\mathbb{C}[SU_n]_q$ on $\mathrm{Pol}(\mathrm{Mat}_n)_q$ given by a $*$-homomorphism

$$\mathcal{D} : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_n)_q \otimes \mathbb{C}[SU_n]_q \otimes \mathbb{C}[SU_n]_q$$

$$\mathcal{D}(z_k^j) = \sum_{a,b=1}^{n} z_a^b \otimes t_{ak} \otimes t_{bj}.$$

In the case $q = 1$, this co-action comes from the two different actions of $SU_n$ on $\mathrm{Mat}_n$

$$(A, X) \mapsto AX \quad (A, X) \mapsto XA^t.$$

for $X \in \mathrm{Mat}_n$ and $A \in SU_n$.

# The Fock representation $\pi_{F,n}$ of $\mathrm{Pol}(\mathrm{Mat}_n)_q$

Vaksman et al proved the following:

**Theorem**

*There exists a unique faithful bounded irreducible $*$-representation $\pi_{F,n} : \mathrm{Pol}(\mathrm{Mat}_n)_q \to B(H_{F,n})$ defined by the property that there exists a vector $v_0 \in H_{F,n}$ s.t*

$$\pi_{F,n}(z_k^j)^* v_0 = 0$$

*for $1 \leq k, j \leq n$.*

The vector $v_0$ is called a *vacuum vector.*

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# Definition of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ again

$$z_a^\alpha z_b^\beta - q z_b^\beta z_a^\alpha = 0, \qquad a = b \ \&\ \alpha < \beta, \text{ or } a < b \ \&\ \alpha = \beta,$$
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# The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$

For $m \le n$ there exists a $*$-homomorphism

$$\mathrm{Pol}(\mathrm{Mat}_m)_q \to \mathrm{Pol}(\mathrm{Mat}_n)_q$$

$$z_k^j \mapsto z_{k+n-m}^{j+n-m}$$

and especially

$$\rho : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_{2n})_q$$

$$z_k^j \mapsto z_{k+n}^{j+n}.$$

# The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_n)_q$

There is a $*$-homomorphism

$$\psi : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathbb{C}[SU_n]_q$$

$$z_k^j \mapsto (-q)^{k-n} t_{kj}.$$

Hence there is a $*$-homomorphism

$$\psi \circ \rho : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathbb{C}[SU_{2n}]_q$$

$$z_k^j \mapsto (-q)^{k-n} t_{(k+n),(j+n)}$$

The idea is to construct a $*$-representation $\Pi$ of $\mathbb{C}[SU_{2n}]_q$ s.t

$$\Pi \circ \psi \circ \rho \cong \pi_{F,n}.$$

# $*$-representations of $\mathbb{C}[SU_n]_q$

We present quickly parts of the representation theory of $\mathbb{C}[SU_n]_q$ due to Soibelman.

If $\pi : \mathbb{C}[SU_2]_q \to B(\ell^2(\mathbb{Z}_+))$ is the $*$-representation given by

$$\pi(t_{11}) = S^* C_q, \quad \pi(t_{12}) = -qD_q,$$
$$\pi(t_{21}) = D_q, \quad \pi(t_{22}) = C_q S$$

where, in the standard orthonormal basis $\{e_m\}_{m=0}^{\infty}$, we have

$$Se_m = e_{m+1}, \quad C_q e_m = \sqrt{1-q^{2m}} e_m, \quad D_q e_m = q^m e_m.$$

Consider the $*$-homomorphisms $\mathbb{C}[SU_n]_q \to \mathbb{C}[SU_2]_q$

$$\phi_i(t_{ii}) = t_{11}, \qquad \phi_i(t_{i+1i+1}) = t_{22},$$
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and $\phi_i(t_{kj}) = \delta_{k,j} I$ otherwise.

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# $*$-representations of $\mathbb{C}[SU_n]_q$

Let $\pi_i : \mathbb{C}[SU_n]_q \to B(\ell^2(\mathbb{Z}_+))$ be the composition $\pi \circ \phi_i$. Let $s_i$ denote the adjacent transposition $(i, i+1)$ in the symmetric group $S_n$.

## Definition

For an element $s \in S_n$ consider a minimal decomposition of $s = s_{j_1} s_{j_2} \ldots s_{j_m}$ into a product of adjacent transposition and let $\pi_s$ be the $*$-representation of $\mathbb{C}[SU_n]_q$ given by $\pi_{j_1} \otimes \pi_{j_2} \otimes \cdots \otimes \pi_{j_m}$.

Up to isomorphism, $\pi_s$ is independent of the specific minimal decomposition. Up to action of torus, all irreducible representation of $\mathbb{C}[SU_n]_q$ arises this way (Soibelman).

# Construction of Fock representation

In $S_{2n}$ let

$$s = \begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ n+1 & n+2 & \dots & 2n & 1 & 2 & \dots & n \end{pmatrix}.$$

## Theorem

*$\pi_s \circ \psi \circ \rho$ is isomorphic to the Fock representation $\pi_{F,n}$ and $\overline{\pi_{F,n}(\mathrm{Pol}(\mathrm{Mat}_n)_q)} \subseteq C^*(S)^{\otimes n^2}$.*

Combining this with the faithfulness of the Fock representation gives

## Corollary

*$\mathrm{Pol}(\mathrm{Mat}_n)_q$ is isomorphic to the $*$-sub-algebra of $\mathbb{C}[SU(2n)]_q$ generated by $\{t_{(k+n)(j+n)}\}$ with $1 \leq k, j \leq n$.*

# Lifting representations of $\mathrm{Pol}(\mathrm{Mat}_n)_q$

**Theorem (G.)**

*For every $*$-representation $\pi$ of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ there is a $*$-representation $\Pi$ of $\mathbb{C}[SU_{2n}]_q$ s.t*

$$\Pi \circ \psi \circ \rho \cong \pi.$$

Generalizes

$$T \in \mathrm{Mat}_n \Rightarrow \begin{bmatrix} T^* & \sqrt{I - T^*T} \\ -\sqrt{I - TT^*} & T \end{bmatrix} \in SU_{2n}.$$

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# Universality of the Fock

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*The universal $C^*$-algebra of* $\mathrm{Pol}(\mathrm{Mat}_n)_q$ *exists and is isomorphic to* $\overline{\pi_{F,n}(\mathrm{Pol}(\mathrm{Mat}_n)_q)} \subseteq C^*(S)^{\otimes n^2}$.
*Every other irreducible representation is a quotient of* $C^*(S)^{\otimes n^2}$.

We denote the universal $C^*$-algebra of $\mathrm{Pol}(\mathrm{Mat}_n)_q$ by $C(\overline{\mathbb{D}_n})_q$

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# Shilov Boundary

The $q$-analogue of holomorphic polynomials is the algebra $\mathcal{A}(Mat_n)_q \subseteq \mathrm{Pol}(\mathrm{Mat}_n)_q$ generated by the holomorphic generators $\{z_k^j\}$.

Consider the ideal $J_n$ in $\mathrm{Pol}(\mathrm{Mat}_n)_q$ generated by the elements:

$n = 1$: $J_n = \langle z_1 z_1^* - 1 \rangle$

$n > 1$: $J_n = \langle \sum_{j=1}^{n} q^{2n-\alpha-\beta} z_j^\alpha (z_j^\beta)^* - \delta^{\alpha,\beta}, \alpha, \beta = 1, 2, \ldots, n \rangle$

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# Algebraic Shilov Boundary Ideal

It was proven by Vaksman that

$$\mathrm{Pol}(\mathrm{Mat}_n)_q/J \cong \mathbb{C}[U_n]_q$$

i.e the name make sense.

Let $\mathcal{A}(\overline{\mathbb{D}_n})_q \subseteq C(\overline{\mathbb{D}_n})_q$ be the closure of $\mathcal{A}(\mathrm{Mat}_n)_q$ and $\bar{J}_n \subseteq C(\overline{\mathbb{D}_n})_q$ be the closure of $J_n$.

Is $\bar{J}_n$ the Shilov boundary ideal for $\mathcal{A}(\overline{\mathbb{D}_n})_q$ relative to $C(\overline{\mathbb{D}_n})_q$?

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# Shilov Boundary

## Theorem

*$\bar{J}_n$ is the Shilov boundary ideal of $\mathcal{A}(\overline{\mathbb{D}})_q$ relative to $C(\overline{\mathbb{D}_n})_q$*

We sketch the proof:

The idea is to first prove

$$\pi_{F,n}(a) = P_H \psi(a)|_H$$

$$\forall a \in \mathcal{A}(\overline{\mathbb{D}_n})$$

where the $*$-representation $\psi$ annihilates $\bar{J}_n$.

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# Sketch of proof

## Theorem (Sz-Nagy)

*Let $T \in B(H)$ with $\|T\| \leq 1$. Then there exists a Hilbert space $K$, $K \supset H$ and a unitary operator $U$ on $K$ such that*

$$T^n = P_H U^n|_H \forall n \geq 0.$$

Consider the case when $n = 1$, the "quantum unit disc" case

We have $\| \pi_{F,1}(z_1^1)\| \leq 1$.

By (Sz-Nagy),

$$\pi_{F,1}(z_1^1)^k = P_H U^k|_H$$

and $z_1^1 \mapsto U$ annihilates $\bar{J}_1$.

This proves the case $n = 1$.

# The induction step

In general, we use induction on $n$.

Assume the result holds for $n-1$.

- For $\varphi \in [-\pi, \pi)$, there are homomorphisms $\Pi_\varphi : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_{(n-1)})_q$
- If $\pi$ is a $*$-representation of $\mathrm{Pol}(\mathrm{Mat}_{n-1})_q$ annihilating $J_{n-1}$, then $\pi \circ \Pi_\varphi$ annihilates $J_n$.
- Recall the co-action

$$\mathcal{D} : \mathrm{Pol}(\mathrm{Mat}_n)_q \to \mathrm{Pol}(\mathrm{Mat}_n)_q \otimes \mathbb{C}[SU_n]_q \otimes \mathbb{C}[SU_n]_q.$$

- If $\pi$ annihilates $J_n$ and $\pi_{w_1}, \pi_{w_2}$ are $*$-representation of $\mathbb{C}[SU_n]_q$, then the $*$-representation

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$$(\pi_{F,n-1} \circ \Pi_{\varphi} \otimes \pi_{w_1} \otimes \pi_{w_2}) \circ \mathcal{D}$$

is a $*$-representation of $\mathrm{Pol}(\mathrm{Mat}_n)_q$.

We can choose $w_1, w_2 \in S_n$ s.t we can use (Sz-Nagy) to show $\forall a \in \mathcal{A}(\mathrm{Mat}_n)$ then $\pi_{F,n}(a)$ is dilation of direct integral of representations

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Now use induction and continuity.

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# Shilov Boundary

We prove now that $\bar{J}_n$ is the Shilov boundary.

$I$ is a boundary ideal for $\mathcal{A}(\overline{\mathbb{D}_n})_q$ s.t $I \supset \bar{J}_n$.

Hence $\forall a \in \mathcal{A}((\mathrm{Mat}_n)_q$

$$\|\pi_{F,n}(a) + I\| = \|\pi_{F,n}(a)\| = \|\pi_{F,n}(a) + \bar{J}_n\|.$$

We show that then
$\|\pi_{F,n}(x) + I\| = \|\pi_{F,n}(x) + \bar{J}_n\|, \forall x \in C(\overline{\mathbb{D}_n})_q \Longrightarrow I = \bar{J}_n.$

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As $\mathrm{Pol}(\mathrm{Mat}_n)_q / J_n$, we have

$$(z_j^i)^* + J_n = (-q)^{i+j-2n} (\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_j^i + J_n \Rightarrow$$

$$\forall x \in \mathrm{Pol}(\mathrm{Mat}_n)_q, \exists k \in \mathbb{N}; (\det_q \mathbf{z})^k x + J_n = a + J_n, a \in \mathcal{A}(\mathrm{Mat}_n)_q.$$

Recall that $(\det_q \mathbf{z})^* \det_q \mathbf{z} + J_n = q^{-(n(n-1))} I + J_n$
Thus

$$\|\pi_{F,n}(x)+I\| = q^{\frac{k(n(n-1))}{2}} \|(\pi_{F,n}(\det_q \mathbf{z})^k x)+I\| = q^{\frac{k(n(n-1))}{2}} \|\pi_{F,n}(a)+I\| =$$

$$q^{\frac{k(n(n-1))}{2}} \|\pi_{F,n}(a) + \bar{J}_n\| = q^{\frac{k(n(n-1))}{2}} \|(\pi_{F,n}(\det_q \mathbf{z})^k x) + \bar{J}_n\| =$$

$$\|\pi_{F,n}(x) + \bar{J}_n\|.$$

# Thank You!