Dixmier traces, heat semi-groups, zeta functions and an application to pseudodifferential operators

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Dixmier traces on von Neumann algebras

Framework

- $(\mathcal{N},\tau)$ semifinite von Neumann algebra with a fixed NSF trace

- τ-mesurable operators: $\mathcal{L}_0 = \Big\{T \in \mathsf{Af}(\mathcal{N}) : \lim\limits_{\lambda\to\infty} \tau\Big(\chi_{[\lambda,\infty)}(|T|)\Big) = 0\Big\}$

- Noncommutative L^p-spaces: $\mathcal{L}^p = \{T \in \mathcal{L}_0 \,:\, \tau(|T|^p) < \infty\}$

- τ-compact operators: $\mathcal{K}_\tau = \overline{\mathcal{F}^{\|\cdot\|}}$ ($\mathcal{F}$ algebra generated by finite trace projections)

- Generalized singular values:

$$\mu(T,t) = \inf_{P=P^*=P^2\in\mathcal{N}} \Big\{\|PT\| \,:\, \tau(1-P) \le t\Big\}, \quad t > 0$$

- Hardy-Littlewood-Pólya majorization:

$$S \prec\prec T \;\Leftrightarrow \int_0^t \mu(S,s)\,ds \le \int_0^t \mu(T,s)\,ds\,, \quad \forall t > 0$$

Examples

- $(\mathcal{N}, \tau) = (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$

$$\mathcal{L}_0 = \mathcal{B}(\mathcal{H})$$
$$\mathcal{K}_\tau = \mathcal{K}(\mathcal{H})$$
$$\mu(T, t) = \sum_{n \geq 0} \mu_n(T) \chi_{(n,n+1]}$$

- $(\mathcal{N}, \tau) = (L^\infty(X, \mu), \int_X . d\mu)$

$$\mathcal{L}_0 = \{f : \mu(\{|f| = \infty\}) < \infty\}$$
$$\mathcal{K}_\tau = \{f : \lim_{t \to \infty} f^*(t) = 0\}$$
$$\mu(f, t) = f^*(t)$$

Decreasing rearrangement: $f^*(t) = \inf\{s \geq 0 : \mu(\{|f| > s\}) \leq t\}$

Symmetric operator spaces

- A Banach subspace $(\mathcal{E}, \|.\|_{\mathcal{E}})$ of $\mathcal{L}_0$ such that

$$\begin{array}{llcl} (i) & \mu(B) = \mu(A) & \Rightarrow & B \in \mathcal{E},\ \|B\|_{\mathcal{E}} = \|A\|_{\mathcal{E}} \\ (ii) & B \prec\prec A & \Rightarrow & B \in \mathcal{E},\ \|B\|_{\mathcal{E}} \leq \|A\|_{\mathcal{E}} \end{array}$$

$\forall A \in \mathcal{E}$, $\forall B \in \mathcal{L}_0$ is called a fully symmetric operator space (FSOS)

Example: $(E, \|.\|_E)$ fully symmetric Banach space on $[0, \infty)$

$$\mathcal{E}(E) = \Big\{ T \in \mathcal{L}_0 \, : \, \mu(T) \in E \Big\}, \quad \|T\|_{\mathcal{E}(E)} = \|\mu(T)\|_E$$

If $\mathcal{N}$ is atomless or $\mathcal{B}(\mathcal{H})$ then every FSOS arise this way

- When $\mathcal{E} = \mathcal{E}(E)$, Boyd index $1 \leq \mathfrak{p}_{\mathcal{E}} \leq \mathfrak{q}_{\mathcal{E}} \leq \infty$

$$\mathfrak{p}_{\mathcal{E}} := \lim_{a \downarrow 0} \frac{\log a^{-1}}{\log \|D_a\|_{E \to E}}, \qquad \mathfrak{q}_{\mathcal{E}} := \lim_{a \to \infty} \frac{\log a^{-1}}{\log \|D_a\|_{E \to E}}.$$

Symmetric functionals

• Fix $\mathcal{E}$ an FSOS. An element $\varphi \in \mathcal{E}^*$ is a fully symmetric functional (FSF) if for all $A, B \in \mathcal{E}_+$

$$\mu(A) = \mu(B) \;\Rightarrow \varphi(A) = \varphi(B) \quad \text{and} \quad A \prec\prec B \;\Rightarrow \varphi(A) \le \varphi(B)$$

Remark: A FSF is a trace (unitarily invariant functional)!

• A FSF φ is singular if $\varphi|_{\mathcal{E}_0} = 0$, where $\mathcal{E}_0 = \overline{\mathcal{F}}^{\|\cdot\|_{\mathcal{E}}}$

• A FSF φ is supported at 0 if $\varphi|_{\mathcal{E} \cap \mathcal{N}} = 0$ (hence singular)

• A FSF φ is supported at ∞ if $\varphi|_{\mathcal{E} \cap \mathcal{L}^1} = 0$ (hence singular)

Lorentz spaces (Marcinkiewicz spaces)

• Let Ω be the set of all increasing concave functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$. Set further

$$\Omega_\infty = \{\psi \in \Omega \ : \ \lim_{t\to\infty} \psi(t) = \infty\}$$

$$\Omega_0 = \{\psi \in \Omega \ : \ \lim_{t\downarrow 0} \frac{t}{\psi(t)} = 0\}$$

$$\Omega_b = \{\psi \in \Omega \ : \ \psi(t) = O(t),\ t \to 0\}$$

• For $\psi \in \Omega$, the operator Lorentz space $\mathcal{M}_\psi$ is the set of $T \in \mathcal{L}_0$ s.t.:

$$\|T\|_{\mathcal{M}_\psi} = \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu(T,s)\, ds < \infty$$

Remark: (i) $\mathcal{M}_\psi$ is an FSOS
(ii) $\mathcal{M}_\psi(\mathbb{R}_+^*) \subset L^\infty(\mathbb{R}_+^*) \Leftrightarrow \psi \in \Omega_b \Rightarrow \mathcal{M}_\psi \subset \mathcal{N}$

Singular trace on Lorentz spaces

Theorem [Dodds-Pagter-Semenov-Sukochev] Let $\psi \in \Omega$
(i) $\mathcal{M}_\psi$ possesses a FSF supported at 0 iff

$$\psi \in \Omega_0 \quad \text{and} \quad \exists a > 1 \ : \ \liminf_{t \downarrow 0} \frac{\psi(at)}{\psi(t)} = 1$$

(i) $\mathcal{M}_\psi$ possesses a FSF supported at ∞ iff

$$\psi \in \Omega_\infty \quad \text{and} \quad \exists a > 1 \ : \ \liminf_{t \to \infty} \frac{\psi(at)}{\psi(t)} = 1$$

Here: only FSF supported at ∞ are considered!

More manageable conditions

$$\exists\, a > 1\,, \qquad \lim_{t\to\infty} \frac{\psi(at)}{\psi(t)} = 1 \tag{1}$$

$$\lim_{t\to\infty} \frac{\psi\Big(t\psi(t)\Big)}{\psi(t)} = 1 \tag{2}$$

$$\forall \alpha > 1\,, \quad \lim_{t\to\infty} \frac{\psi(t^\alpha)}{\psi(t)} \quad \text{exists} \tag{3}$$

- (3) $\Rightarrow$ (2) $\Rightarrow$ (1)

- If (3) holds then

$$\lim_{t\to\infty} \frac{\psi(t^\alpha)}{\psi(t)} = \alpha^{k(\psi)} \quad \text{where} \quad 0 \le k(\psi) = \log\left(\lim_{t\to\infty} \frac{\psi(t^e)}{\psi(t)} \right)$$

Examples

$$\psi_{n,\beta}(t) = \Big(\log(1+\log(1+\cdots\log(1+t^{1/\beta})\ldots))\Big)^{\beta}$$
$$\tilde{\psi}_{n,\beta}(t) = \begin{cases} C^{-1}\,t\,, & t \in [0,C] \\ \exp\Big\{\Big(\log(\ldots\log(t))\ldots\Big)^{\beta}\Big\}\,, & t > C \end{cases}$$

- $\psi_{n,\beta}(t) \in \Omega_b$ satisfies (3) $\forall n \in \mathbb{N}^*$, $\beta > 0$
- $\tilde{\psi}_{n,\beta}(t) \in \Omega_b$ satisfies (3) $\forall n \in \mathbb{N} \setminus \{0,1\}$, $\beta \in (0,1)$, not for $n = 1$
- $\tilde{\psi}_{1,\beta}(t) \in \Omega_b$ satisfies (2) $\forall \beta \in (0,1/2)$, not for $\beta \in [1/2,1)$
- $\tilde{\psi}_{1,\beta}(t) \in \Omega_b$ satisfies (1) $\forall \beta \in (0,1)$

ψ-compatible states

• A state $\omega \in \left(L^\infty(\mathbb{R}^*_+)\right)^*$ is T-invariant, D-invariant or E-invariant if for all $a > 0$, we have

$$\omega \circ T_a = \omega\,, \quad \omega \circ D_a = \omega \quad \text{or} \quad \omega \circ E_a = \omega$$

where

$$T_a f(t) = f(t+a)\,, \quad D_a f(t) = f(at)\,, \quad E_a f(t) = f(t^a)$$

• T-invariant, D-invariant and E-invariant states are singular: they vanish on $C_0(\mathbb{R}^*_+)$

• Let $\psi \in \Omega_\infty$. Then a state $\omega \in \left(L^\infty(\mathbb{R}^*_+)\right)^*$ is ψ-compatible if

$$\exists a > 1 \ : \ \omega\left(\left[t \mapsto \frac{\psi(at)}{\psi(t)}\right]\right) = 1 \quad \Leftrightarrow \quad \omega\left(\left[t \mapsto \frac{t\psi'(t)}{\psi(t)}\right]\right) = 0$$

Dixmier traces and ψ-compatible states

Theorem [Dixmier, Kalton-Sedaev-Sukochev] Let $\psi \in \Omega_\infty$

(i) If ω is a ψ-compatible and D-invariant state, then the map

$$\tau_{\psi,\omega} : \mathcal{M}_\psi^+ \to \mathbb{R}_+ \,, \qquad T \mapsto \omega\Big(\Big[t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T,s)\, ds\Big]\Big)$$

is positively additive and its linear extension is a FSF supported at ∞ and is called a Dixmier trace

(ii) Reciprocally, every normalized FSF supported at ∞ is a Dixmier trace $\tau_{\psi,\omega}$ where ω is a ψ-compatible and D-invariant state

Remarks

• The notion of ψ-compatibility is manageable:
(i) There exists a D-invariant and ψ-compatible state iff $\mathfrak{p}_{\mathcal{M}_\psi} = 1$.
(ii) Every D-invariant state is ψ-compatible iff $\mathfrak{p}_{\mathcal{M}_\psi} = \mathfrak{q}_{\mathcal{M}_\psi} = 1$

• One doesn't need to take a D-invariant and ψ-compatible state to get a Dixmier trace:
If ω is E-invariant and $\psi \in \Omega_\infty$ is such that

$$\lim_{\alpha \downarrow 1} \omega\left(\left[t \mapsto \frac{\psi(t^\alpha)}{\psi(t)}\right]\right) = 1$$

then $\tau_{\psi,\omega}$ is a normalized FSF supported at ∞. Hence there exists a D-invariant and ψ-compatible state ω' such that $\tau_{\psi,\omega} = \tau_{\psi,\omega'}$

How to compute a Dixmier trace?

ζ-functions and heat semi-group

Theorem: [Connes etc] For ω an E-invariant state, we have for all $T \in \mathcal{M}^+_{\text{log}}$ and $B \in \mathcal{N}$:

$$\tau_{\text{log},\omega}(BT) = \omega\Big(\Big[r \mapsto \frac{\zeta_B(T, 1 + \log(r)^{-1})}{\log(1+r)}\Big]\Big)$$
$$= \omega\Big(\Big[\lambda \mapsto \frac{1}{\log(1+\lambda)} \int_0^\lambda \xi_B(T, t^{-1}) \frac{dt}{t^2}\Big]\Big)$$

where

$$\zeta_B(T,z) := \tau(BT^z) \quad \text{and} \quad \xi_B(T,t) := \tau\Big(Be^{-tT^{-1}}\Big)$$

Statement of our problem

What is the class of functions $\psi \in \Omega$ for which

$$\tau_{\psi,\omega}(BT) = C_\zeta(\psi)\,\omega\Big(\Big[r \mapsto \frac{\zeta_B(T, 1+\log(r)^{-1})}{\psi(r)}\Big]\Big)$$
$$= C_\xi(\psi)\,\omega\Big(\Big[\lambda \mapsto \frac{1}{\psi(\lambda)}\int_0^\lambda \xi_B(T, t^{-1})\,\frac{dt}{t^2}\Big]\Big)$$

for all $(\mathcal{N}, \tau)$, $T \in \mathcal{M}_\psi(\mathcal{N}, \tau)^+$, $B \in \mathcal{N}$, ω E-invariant state, and for certain positive constants $C_\zeta(\psi)$ and $C_\xi(\psi)$ which depend on ψ only?

Main result

Theorem:

The answer of our question is positive for all $\psi \in \Omega_\infty \cap \Omega_b$ satisfying condition (3) and such that

$$\mathcal{M}_\psi = \mathcal{L}_\psi = \left\{ T \in \mathcal{L}_0 \ : \ \sup_{p>1} \frac{\|T\|_p}{\psi(e^{(p-1)^{-1}})} < \infty \right\} \tag{4}$$

and in this case we have

$$C_\zeta(\psi) = \Gamma\Big(1 + k(\psi)\Big)^{-1} \qquad \text{and} \qquad C_\xi(\psi) = 1$$

Extrapolation of the field $(\mathcal{L}^p)_{p\geq 1}$

An extrapolation functor

• For all $T \in \mathcal{L}_0$ define

$$\eta_T := \left[p \in (1,\infty) \mapsto \|T\|_p\right]$$

• Given F a fully symmetric space of functions on $(1,\infty)$, define

$$\mathfrak{L}_F := \left\{T \in \mathcal{L}_0 \,:\, \|\eta_T\|_F < \infty\right\}$$

• The correspondence $F \to \mathfrak{L}_F$ defines a functor from the category fully symmetric function spaces on $(1,\infty)$ to the category fully symmetric operator spaces on semifinite von Neumann algebra

Definition: A FSOS $\mathcal{E}$ is called an extrapolated space if there exists F, a fully symmetric space of functions on $(1,\infty)$, such that $\mathcal{E} = \mathfrak{L}_F$ with equivalent norms. Let χ_1 the category of extrapolated space close to $\mathcal{L}^1$ (contained in all $\mathcal{L}^p$ except $\mathcal{L}^1$)

The upper and lower extrapolations

• For $\psi \in \Omega_\infty$, we set We then let

$$F^{\psi} = \left\{ f : (1,\infty) \to \mathbb{C} \ : \ \|f\|_{F^{\psi}} = \sup_{p>1} \frac{|f(p)|}{\|\psi'\|_p} < \infty \right\}$$

$$F_{\psi} = \left\{ f : (1,\infty) \to \mathbb{C} \ : \ \|f\|_{F_{\psi}} = \sup_{p>1} \frac{|f(p)|}{\psi(e^{(p-1)^{-1}})} < \infty \right\}$$

• Accordingly, we let $\mathfrak{L}^{\psi} \equiv \mathfrak{L}_{F^{\psi}}$ and $\mathfrak{L}_{\psi} \equiv \mathfrak{L}_{F_{\psi}}$ the associated extrapolated operator spaces

Proposition: Let $\psi \in \Omega_\infty$. Then $\mathfrak{L}_{\psi} \subset \mathcal{M}_{\psi} \subset \mathfrak{L}^{\psi}$ with

$$\|.\|_{\mathfrak{L}^{\psi}} \leq \|.\|_{\mathcal{M}_{\psi}} \leq \max\left\{ e, \frac{\psi(1)}{\psi'(1)} \right\} \|.\|_{\mathfrak{L}_{\psi}}$$

Conditions of equality

Proposition: Let $\psi \in \Omega_\infty$. Then

$$(i) \qquad \mathcal{M}_\psi = \mathfrak{L}^\psi \quad \Leftrightarrow \quad \mathcal{M}_\psi \in \mathcal{X}_1 \quad \Leftrightarrow \quad \psi(t) \asymp \left(\sup_{p>1} \frac{t^{1/p-1}}{\|\psi'\|_p} \right)^{-1}$$

$$(ii) \qquad \mathcal{M}_\psi = \mathfrak{L}_\psi \quad \Leftrightarrow \quad \psi' \in \mathfrak{L}_\psi(\mathbb{R}_+) \quad \Leftrightarrow \quad \|\psi'\|_p \leq C\,\psi(e^{(p-1)^{-1}})$$

Proposition: Let $\psi \in \Omega_\infty \cap \Omega_b$ such that $\forall \delta > 0$ $[t \mapsto t^{-\delta}\psi(t)]$ is decreasing and that $\exists \rho > 0$ $[t \mapsto \psi(\exp(t^\rho))]$ is concave. Then $\mathfrak{L}_\psi = \mathcal{M}_\psi = \mathfrak{L}^\psi$

Remark: As condition (4) is $\mathcal{M}_\psi = \mathfrak{L}_\psi \Leftrightarrow \mathfrak{L}_\psi = \mathcal{M}_\psi = \mathfrak{L}^\psi$

Application to pseudodifferential operators in $\mathbb{R}^n$

Hörmander-Weyl pseudodifferential calculus

- Weyl quantization map:

$$\mathrm{OP}_W \in \mathcal{L}\Big(\mathcal{S}'(\mathbb{R}^{2n}), \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)\big)\Big)$$

$$\big(\mathrm{OP}_W(T)\phi\big)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\xi.(x-y)}\, T\Big(\frac{x+y}{2}, \xi\Big)\, \phi(y)\, d^n y\, d^n\xi$$

- Properties (not shared by variants):

$$\mathrm{OP}_W(\overline{T}) = \mathrm{OP}_W(T)^* \qquad \mathrm{OP}_W \in \mathcal{U}\Big(L^2(\mathbb{R}^{2n}), \mathcal{L}^2\big(L^2(\mathbb{R}^n)\big)\Big),$$

- Hörmander symbol classes $\mathcal{S}(m,g)$: g slowly varying, σ-temperate metric on $\mathbb{R}^{2n}$ and $m : \mathbb{R}^{2n} \to \mathbb{R}_+^*$ g-continuous, (σ, g)-temperate weight function satisfying $m \in S(m,g)$

- Planck function

$$h_g^2(x,\xi) = \sup_{(t,\tau)\in\mathbb{R}^{2n}} \frac{g_{x,\xi}(t,\tau)}{g^\sigma_{x,\xi}(t,\tau)}$$

Basic properties [Hörmander]

- Trace inequality:

$$\|\mathrm{OP}_W(f)\|_1 \leq C_k(\|f\|_1 + \|h_g^k m\|_1 \|f\|_{k;m,g})$$

- Continuity of the composition product:

$$\star : S(m_1, g) \times S(m_2, g) \to S(m_1 m_2, g)$$

- Asymptotic behavior of the composition product:

$$f_1 \star f_2 - f_1 f_2 \in S(m_1 m_2 h_g, g)$$

More properties [Nicola-Rodino]

Suppose further that $m(x,\xi) \to 0$, $(x,\xi) \to \infty$:

- $\mathrm{OP}_W(c+m^{-1})$ is positive and invertible with inverse in $\mathrm{OP}_W(\mathcal{S}(m,g))$

- For all $t \geq 0$, we have

$$e^{-t\,\mathrm{OP}_W(c+m^{-1})} = \mathrm{OP}_W(b_t) + S_t$$

with $\{b_t\}_{t>0}$ a bounded family of symbols in $S(1,g)$ and $\|S_t\|_1 \leq C\,t$

- For fixed $t > 0$, $b_t \in \cap_{l\in\mathbb{N}} S(m^l, g)$

- b_t can be written as

$$b_t = e^{-t(c+m^{-1})} + \sum_{j=1}^{N} b_{t,j} \quad \text{with} \quad |b_{t,j}| \leq C_j\, e^{-t(c+m^{-1})/2}\, h_g^j$$

Statement of the result

Theorem: Let $\psi \in \Omega_\infty$ satisfying (2), $m \in \mathcal{M}_\psi(\mathbb{R}^{2n})$ and assume that $h_g \in L^q(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$, then:

(i) $\mathsf{OP}_W : \mathcal{S}(m,g) \to \mathcal{M}_\psi\big(L^2(\mathbb{R}^n)\big)$ continuously

(ii) If $\psi \in \Omega_\infty$ satisfies further (3) and (4), then for $f \in \mathcal{S}(m,g)$ and an E-invariant state ω:

$$\mathsf{Tr}_{\psi,\omega}\big(\mathsf{OP}_W(f)\big) = \int_{\psi,\omega} f$$

$$= \frac{1}{\Gamma(1+k_\psi)}\,\omega\left(\left[r \mapsto \frac{1}{\psi(r)} \int_{\mathbb{R}^{2n}} f(x,\xi)\,|f(x,\xi)|^{1/\log(r)}\,d^n x\, d^n \xi\right]\right)$$

Comments

• Nicely complement the classical relation

$$\mathrm{Tr}\Big(\mathrm{OP}_W(f)\Big) = \int f$$

• Extends the results of Nicola-Rodino, where

i) No formula for the Dixmier trace is given

ii) $\psi(t) = \log(1+t)$

iii) $T \in \mathcal{L}^{1,\infty} \subsetneq \mathcal{M}_{\log}$

iv) $h_g(x,\xi) \le C(1+|x|+|\xi|)^{-\varepsilon}$

• Generalizable to pseudodifferential operators on manifolds of bounded geometry with symbols in the Shubin classes