Particle Beam Models in Radiation Oncology

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Outline

- Continuous problems:
  Asymptotic expansions for the transport equation $\Longrightarrow$
  - The Fokker-Planck development/Fermi equation.
    Spencer's method of moment $\Longrightarrow$
  - bipartition models for neutral and charged particle transport.
- Numerical Methods:
  - Monte Carlo method
  - Finite elements
  - Finite differences/Lax- Wendroff
- Open Problems:
  - (Multi)- Energy dependent cases, Inhomogeneous media.

The general transport equation

The flow of particles through the background medium is described by:

$$\left\{ \frac{1}{v}\frac{\partial\psi(E,\Omega)}{\partial t} + \Omega\cdot\nabla\psi(E,\Omega) + \sigma(E)\psi(E,\Omega) = \underbrace{\int_0^\infty dE' \int_{4\pi} d\Omega' \overbrace{\sigma_s(E',E,\Omega'\cdot\Omega)}^{\text{inscattering kernel}} \psi(E',\Omega)}_{:=I} + \underbrace{q(E,\Omega)}_{\text{source}} . \right.$$

- $\sigma(E) = \sigma_a(E) + \sigma_s(E)$;   total cross section
- $\psi(E,\Omega) = vf(E,\Omega)$;   ($v$ is the particle speed), current.
- $\psi(E,\Omega) = \psi(\mathbf{r},E,\Omega,t),\ldots,$

Transport problems of electrons and other charged particles are relevant, for example, in radiation treatment planning: calculation of radiation doses.

Direct numerical approach/ Modified scattering kernel

- Deterministic methods: Very difficult since the mesh size must be on the same scale as the mean free path, the average distance a particle travels between scattering events, which is very small. This implies an unrealistically fine degree of numerical resolution.
- Monte-Carlo simulation: Very time consuming since a very large number of scattering interactions must be followed for each particle before it demise by either absorption or leakage out of the system.

To circumvent these difficulties, it has been suggested to

- Replace the integral scattering operator with a differential Fokker-Planck operator, Chandrasekhar[43], Rosenbluth [57]. The effect of this replacement is that the dominant (large) in and out scattering terms cancel, thus efficiently increasing the mean free path.
- Separate the large-angle (diffusion) scattering particles from the small-angle (straightforward) scattered ones: bipartition model based on Spencer's[59], moment method.

Mono-energetic Pencil Beam Problem

- Transport (exact), $\sigma_t = \sigma_{s0},\ \sigma_{tr} = \sigma_{s0} - \sigma_{s1}, \ldots.$

$$\begin{cases} \mu\dfrac{\partial\psi}{\partial x} + \eta\dfrac{\partial\psi}{\partial y} + \xi\dfrac{\partial\psi}{\partial z} + L\sigma_t\psi(\mathbf{r},\Omega) = \displaystyle\int_{4\pi} d\Omega'\,\sigma_s(\Omega'\cdot\Omega)\psi(\mathbf{r},\Omega) \\ (B.C.1) \qquad \psi|_{x=0} = \delta(y)\delta(z)\dfrac{\delta(\mu-1)}{2\pi}, \quad 0 < \mu \le 1. \\ (B.C.2) \qquad \psi|_{x=1} = 0, \qquad -1 \le \mu < 0. \end{cases}$$

- Fokker-Planck

$$\begin{cases} \mu\dfrac{\partial\psi}{\partial x} + \eta\dfrac{\partial\psi}{\partial y} + \xi\dfrac{\partial\psi}{\partial x} = \dfrac{L\sigma_{tr}}{2}\Big[\dfrac{\partial}{\partial\mu}(1-\mu^2)\dfrac{\partial}{\partial\mu} + \dfrac{1}{(1-\mu^2)}\dfrac{\partial^2}{\partial\theta^2}\Big]\psi(\mathbf{r},\Omega) \\ (B.C.1) \qquad \& \qquad (B.C.2) \end{cases}$$

- Fermi $$\begin{cases} \mu\dfrac{\partial\psi}{\partial x} + \eta\dfrac{\partial\psi}{\partial y} + \xi\dfrac{\partial\psi}{\partial x} = \dfrac{L\sigma_{tr}}{2}\Big[\dfrac{\partial^2}{\partial\eta^2} + \dfrac{\partial^2}{\partial\xi^2}\Big]\psi(\mathbf{r},\Omega) \\ \psi|_{x=0} = \delta(y)\delta(z)\delta(\eta)\delta(\xi). \end{cases}$$

Fermi equation in a slab of thickness L

$x \in I_x := [0, L]$, $I_\perp := I_y \times I_z := [-y_0, y_0] \times [-z_0, z_0]$:

$$\begin{cases} J_x + zJ_y = \varepsilon J_{zz}, & \text{in } \Omega = I_x \times I_\perp, \\ J_z(x, y, \pm z_0) = 0, & \text{for } (x, y) \in I_x \times I_y, \\ J(0, x_\perp) = f(x_\perp), & \text{for } x_\perp \in I_\perp, \\ J(x, x_\perp) = 0, & \text{on } \Gamma^-_{\tilde{\beta}} = \{(x, x_\perp) \in \Gamma := \partial\Omega,\ \tilde{n} \cdot \tilde{\beta} < 0\}, \end{cases}$$

$\tilde{\beta} = (1, z, 0)$ and $\tilde{n}$ is the outward unit normal to Γ at $(x, x_\perp) \in \Gamma$.

This equation is interpreted as:

- time-dependent (x viewed as time variable),
- degenerate (convection in y, diffusion in z),
- convection dominated (ε is small),
- convection-diffusion problem.

Principles of bipartition model/Lewi's electron transport

Monoenergetic monodirectional source in an infinite homogeneous slab

$$-\frac{\partial f}{\partial \tau} + \mu \frac{\partial f}{\partial x} = \int_{4\pi} \frac{N_A R_0}{A} \sigma(\tau, \mathbf{u} \cdot \mathbf{u}')[f(x, \mu', \tau) - f(x, \mu, \tau)] d\mathbf{u}' + \frac{1}{2\pi} \delta(1-\mu)\delta(1-\tau)\delta(x) := C_f + \frac{1}{2\pi}\delta(1-\mu)\delta(1-\tau)\delta(x)$$

split the solution as: $f(x, \mu, \tau) = f_s(x, \mu, \tau) + f_d(x, \mu, \tau)$

- f_s: the fluence of straightforward electrons satisfies
$$-\frac{\partial f_s}{\partial \tau} + \mu \frac{\partial f_s}{\partial x} = C_{f_s} - S_{diff}(x, \mu, \tau) + \frac{1}{2\pi}\delta(1-\mu)\delta(1-\tau)\delta(x),$$
- f_d: the fluence of the diffusion electrons satisfies
$$-\frac{\partial f_d}{\partial \tau} + \mu \frac{\partial f_d}{\partial x} = C_{f_d} + S_{diff}(x, \mu, \tau)$$
- spherical harmonics exp. $\Longrightarrow$: $f_s = \sum_{l=0}^{\infty} \frac{2l+1}{2\pi} P_l(\mu) A_l(\tau, x),$

$$f_d = \sum_{l=0}^{\infty} \frac{2l+1}{2\pi} P_l(\mu) N_l(\tau, x), \quad S_{diff} = \sum_{l=0}^{m} \frac{2l+1}{2\pi} P_l(\mu) S_l(\tau, x).$$

Principles of bipartition method (cont.)

- Blanchard's and Spencer's approximation formula $\Longrightarrow C_f$.
- Partition condition:
  $S_{diff}(x, \mu_i, \tau) = C_f(x, \mu_i, \tau), \quad i = 0, 1, \ldots, m \Longrightarrow S_l(\tau, x)$.
  - Here, $\mu_i (i = 0, 1, \ldots, m)$ is a set of $m + 1$ different large-angle directional cosine that are selected arbitrary to some extent.
  - If the straightforward electrons undergo large elastic scattering they will be removed from the straightforward-electron group. They are treated as a source of diffusion electrons.
- Convection: "Mean electron track" for straightforward electrons +Fourier transform $\Longrightarrow A_l(\tau, x)$.
- Diffusion: P_n approximation + Lax-Wendroff $\Longrightarrow N_l(\tau, x)$.

Some current challenges

- To work out an approach for heavy ion particles.
- Access to experimental data relevant in clinical treatment.
- “Computations” comparing the results of different numerical methods.