Chapter 9

Continuous-Variable Approach to Quantum Information

In the Continuous-Variable (CV) approach to quantum information processing, relevant observables are characterised by a continuous spectrum, such as the amplitude \( \hat{q} = (\hat{a} + \hat{a}^\dagger) / (\sqrt{2}) \) and phase \( \hat{p} = (\hat{a} - \hat{a}^\dagger) / (i\sqrt{2}) \) quadratures of the electromagnetic field, satisfying \( [\hat{q}, \hat{p}] = i \). The associated Hilbert space is infinite-dimensional, and the (infinite) energy levels are eigenstates of the number operator \( \hat{n} = \hat{a}^\dagger \hat{a} \). This is opposed to the traditional Discrete Variable (DV) approach, where observable have a discrete spectrum, and the Hilbert space is finite-dimensional. Generally speaking, CV systems offer the advantage that the resource states (e.g. squeezed states or large cluster states, that we will introduce in the following) can be deterministically produced. Moreover, new methods for experimental implementations, offering solutions to scalability, have emerged in the context of CV. For instance, A. Furusawa (Tokyo, Japan) and O. Pfister (Charlottesville, USA) have been able to produce CV entangled states of up to one-million optical modes; in the experiments of N. Treps (Paris, France) several squeezed states are simultaneously available in the same optical cavity. With microwave cavities coupled to superconducting circuits, the Yale group has demonstrated that it is possible to store quantum information for a longer time in a radiation state (namely an encoded “cat state”) then if the corresponding state is encoded in a qubit. CV are therefore promising for the implementation of scalable and robust architectures for quantum computing.

A review of the formalism underlying CV quantum information, namely the quantization of electromagnetic radiation, is provided in Appendix A. In the following, we are going to introduce the basic CV quantum operations, and to learn about several quantum computation models and protocols in CV.

9.1 Quantum computing with continuous variables

9.1.1 First definitions, elementary operations and universal gate-sets

We start by introducing a notion of computational universality in CV, as well as the elementary CV operations.

Definition of CV universality (1)

The first definition of computational universality in CV that we will encounter in this course (and the first one historically) is the following: a CV QC system is universal if it can simulate the action of any Hamiltonian
\(e^{iH(\hat{p}_i, \hat{q}_j)}\) consisting of a polynomial of the quadratures \(\hat{q}_i\) and \(\hat{p}_j\) in each mode \(i, j\), to an arbitrary fixed accuracy [Lloyd and Braunstein, 1999, Gu et al., 2009].

It is also convenient to introduce the notion of Gaussian universality. This is given by LUBOs operations, for Linear unitary Bogoliubov transformations. They consist of any evolution operator involving at most quadratic polynomial in \(\hat{q}\) and \(\hat{p}\). Introducing \(\hat{x} = (\hat{q}_1, \hat{q}_2...\hat{q}_N, \hat{p}_1, \hat{p}_2...\hat{p}_N)^T \equiv (\vec{q}, \vec{p})^T\), then these operations can be represented as

\[\hat{U}^\dag \hat{x} \hat{U} = S\hat{x} + \hat{c}, \quad \text{(9.1)}\]

with \(S\) a symplectic matrix, \(\hat{c}\) a displacement (see e.g. Ref. [Menicucci et al., 2011]). In the following, we are going to list the relevant CV elementary Gaussian operations, which allows reaching Gaussian universality:

**Single mode gaussian transformations**

a) Rotations: \(R(\theta) = e^{i\theta(a^2 + b^2)}\)\(^1\). Particular example: Fourier transform \(R(\pi/2) = F\), with\(^2\)

\[F|s\rangle_q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dr e^{irs} |r\rangle_q = |s\rangle_p \]

\[F^\dag|s\rangle_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dr e^{-irs} |r\rangle_p = |s\rangle_q \quad \text{(9.4)}\]

with \(|r\rangle_q = \frac{1}{\sqrt{2\pi}} e^{irs}\) and \(|s\rangle_p, |s\rangle_q\) the eigenstates of the quadrature operators,

\[q|s\rangle_q = s|s\rangle_q \]

\[\hat{p}|s\rangle_p = s|s\rangle_p. \quad \text{(9.5)}\]

In particular, \(|0\rangle_p\) is the infinitely p-squeezed state.

b) Quadrature displacements:

\[X(s) = e^{-is\hat{p}}, \quad \text{such that} \quad X(s)|r\rangle_q = |r + s\rangle_q \]

\[Z(s) = e^{is\hat{q}}, \quad \text{such that} \quad Z(s)|r\rangle_p = |r + s\rangle_p. \quad \text{(9.6)}\]

c) Squeezing: \(S(s) = e^{-is(\hat{q}\hat{p} + \hat{p}\hat{q})}\)

d) Shear: \(D_{2,q} = e^{is^2/2}\)

Any single mode gaussian operation can be decomposed in a) rotations; b) quadrature displacement; c) or d), i.e. squeezing or shear. I.e.

\[\{D_{1,q}(s) = Z(s), D_{2,q}(s), F = R(\pi/2)\} \quad \text{universal set for single-mode gaussian operations}\]

\(^1\)Note that the action of a rotation of the quadratures of a single mode looks like

\[R(\theta) \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix}, \quad \text{(9.2)}\]

not to be confused with an equivalent matrix acting on the two-mode amplitude quadratures vector \(\begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix}\), realising the rotation of one quadrature \(\hat{q}_1\) with respect to another one \(\hat{q}_2\).

\(^2\) The corresponding rotation of the second mode quadrature is expressed by

\[\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix} \quad \text{(9.3)}\]

which corresponds to the transformations \(F^\dag \hat{q} F = -\hat{p}\) and \(F^\dag \hat{p} F = \hat{q}\), i.e. \(F^\dag \hat{a} F = i\hat{u}\) and \(F^\dag \hat{a}^\dagger F = -i\hat{a}^\dagger\).
Multimode gaussian transformations

The addition of any non trivial two-mode gaussian gate, such as the $CZ$ interaction $e^{i\hat{q}_1 \hat{q}_2}$, combined with the set of single-mode operations above allows for any general multi-mode gaussian operation (see Ref. [Menicucci et al., 2011] for comments on the generalization to imperfect gaussian operations and gaussian measurements).

$$\{D_{1,q}(s) = Z(s), D_{2,q}(s), F = R(\pi/2), CZ\} \text{ universal set for multi-mode gaussian operations}$$

They are enough to perform some algorithms such as error correcting code regarding errors on single channels (though, non-gaussian measurements are required to correct errors such as loss on all channels simultaneously). Yet, all the algorithm implying only gaussian unitaries acting on Gaussian states could be efficiently simulated on a classical computer, as we are going to see later [Mari and Eisert, 2012].

Single-mode universal transformations

For a single mode, all gaussian operations together with any non-gaussian operation provide universality. For example, the set $D_{k,q} = e^{i\hat{q}_k}$ for $k = 1, 2, 3$ for all $s \in R$ together with $F$ provides universal single mode quantum computation (i.e., can be used to implement any single-mode unitary operation to arbitrary fixed accuracy):

$$\{D_{1,q}(s) = Z(s), D_{2,q}(s), D_{3,q}(s), F = R(\pi/2)\} \text{ universal set for single-mode q.c.}$$

Multi-mode universal transformations

Adding to this any non-trivial two-mode interaction provide universal quantum computation [Lloyd and Braunstein, 1999]. I.e.

$$\{D_{1,q}(s) = Z(s), D_{2,q}(s), D_{3,q}(s), F = R(\pi/2), CZ\} \text{ universal set for multi-mode q.c.}$$

$$\{e^{i\hat{q}_1}, e^{i\hat{q}_2}, e^{i\hat{p}_1/2}, e^{i\hat{q}_1/2}, e^{i\hat{q}_3}\} \text{ universal set for multi-mode q.c.}$$

9.2 Measurement-based quantum computation: the general paradigm in CV

CV quantum computation finds its most natural formulation within the measurement-based model. The reason for this is the availability of massively large cluster states, that can be deterministically generated (see experimental results by Furusawa, Pfister and Treps). In this framework, we can reformulate the notion of universal quantum computation introduced in Sec.9.1.1 as follows: for any CV unitary $U = e^{iH(\hat{q}, \hat{p})}$ (corresponding to an arbitrary polynomial of $\hat{q}_i$ and $\hat{p}_j$) and any given input $|\phi\rangle$ there exists an appropriate graph state such that by entangling the graph state locally with $|\phi\rangle$ and applying an appropriate sequence of single-mode measurements, $U|\phi\rangle$ is computed. We are now going to retrace all the steps seen in Sec.3.2 when we have introduce the measurement-based quantum computation model for qubits, but here in the framework of CV. We start with the definition of CV cluster states.

9.2.1 Cluster states in Continuous Variables

Ideal cluster states are defined as follows: given a graph with $N$ vertices and a certain number of edges relying these vertices according to a specific structure modeled by an adjacency matrix $V$, a CV cluster
state is obtained starting from a collection of $N$ infinitely p-squeezed states, and applying $C_Z$ interactions according to the graph structure, i.e. the controlled-Z gate $e^{i\hat{q}_1\hat{q}_2}$, yielding

$$|\psi_V\rangle = \hat{C}_Z[V]|0\rangle^\otimes N = \prod_{j,k} e^{i\hat{V}_{j,k}\hat{q}_j\hat{q}_k}|0\rangle^\otimes N = e^{i\hat{q}^T\hat{V}\hat{q}}|0\rangle^\otimes N \quad (9.7)$$

Here $V$ is a real and symmetric matrix, with finite elements.

Eq.(9.7) implies that a cluster state satisfies a nullifier relation, as detailed here below. Let us first introduce the following definition: If an operator $\hat{K}$ satisfies for a state $|\phi\rangle$

$$K|\phi\rangle = |\phi\rangle \quad (9.8)$$

we call it a "stabilizer" for the state $|\phi\rangle$. Eq.(9.8) implies that

$$UKU^\dagger(U|\phi\rangle) = U|\phi\rangle, \quad (9.9)$$

i.e that $UKU^\dagger$ stabilizes $U|\phi\rangle$. Note furthermore that

$$e^{-is\hat{p}}|0\rangle_p = X(s)|0\rangle_p = |0\rangle_p \forall s. \quad (9.10)$$

From Eqs.(9.10) and (9.9) it follows that the cluster state (9.7) is stabilized by the set

$$K_i = \hat{C}_Z[V]|X_i(s)e^{i\hat{q}^T\hat{V}\hat{q}} = \prod_{j,k} e^{i\hat{V}_{j,k}\hat{q}_j\hat{q}_k} e^{-is\hat{p}_i} e^{-is\hat{V}_{i,m}\hat{q}_m} \quad (9.11)$$

for each $i$. Using that $e^{i\hat{q}_1\hat{q}_2}\hat{p}_1 e^{-i\hat{q}_1\hat{q}_2} = \hat{p}_1 - \hat{q}_2$, we finally obtain that

$$K_i = e^{-is\hat{p}_i} \prod_k V_{i,k} e^{is\hat{q}_k} = X_i(s) \prod_k V_{i,k} Z_k(s). \quad (9.12)$$

Eq.(9.12) is formally equivalent to its analog in the discrete variable case (see Eq.(20.11) in Ref.[D. Druss and G. Leuchs, "Lectures on Quantum Information", Wiley-VCH (2007)]). The $K_i$ form a group. The generators of the corresponding algebra are found by derivation since $K_i = e^{-is\hat{H}_i}$. Note that because of Eq.(9.8) it follows that $H_i|\psi_V\rangle = 0$, i.e. $\forall i$. Hence the $H$ operators are called "nullifiers" for the state $|\psi_V\rangle$. They can be easily calculated as

$$H_i = \frac{d K_i}{ds} \bigg|_{s=0} = \frac{d}{ds} \left[ e^{-is\hat{p}_i} \prod_k V_{i,k} e^{is\hat{q}_k} \right] \bigg|_{s=0}$$

$$= \hat{p}_i + i e^{-is\hat{p}_i} \sum_k V_{i,k} i\hat{q}_k \prod_l e^{i\hat{q}_l} \bigg|_{s=0} = \hat{p}_i - \sum_k V_{i,k} \hat{q}_k \quad (9.13)$$

from which follows that

$$\left( \hat{p}_i - \sum_k V_{i,k} \hat{q}_k \right) |\psi_V\rangle = 0. \quad (9.14)$$

From Eq.(9.14) follows immediately that

$$\langle\psi_V|\Delta^2 \left( \hat{p}_i - \sum_k V_{i,k} \hat{q}_k \right) |\psi_V\rangle = 0, \quad (9.15)$$

which for states with zero average also reads $\langle\psi_V| (\hat{p}_i - \sum_k V_{i,k} \hat{q}_k)^2 |\psi_V\rangle = 0$. 

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9.2.2 The CV MBQC paradigm

Consider the scheme in Fig. 9.1.

\[ |\phi\rangle \quad \rightarrow \quad X(m)F|\phi\rangle \]
\[ |0\rangle_p \quad \rightarrow \quad D_q \quad \hat{p} = D_q - \hat{p} = m \]

Figure 9.1

- **1) Preparation:** One mode contains the initial state that we want to process \( |\phi\rangle = \int dsf(s) |s\rangle_q \); the other mode is initialized to \( |0\rangle_p \). The input state is hence \( |\phi\rangle \otimes |0\rangle_p = \int dsf(s) |s\rangle_q |0\rangle_p \). Apply a \( C_Z \) gate between the two, obtaining

\[
C_Z(|\phi\rangle \otimes |0\rangle_p) = \int dsf(s) e^{i\hat{q} \otimes \hat{p}} |s\rangle_q |0\rangle_p = \int dsf(s) e^{i\hat{q} \hat{D}_q} |s\rangle_q |0\rangle_p = \int dsf(s) |s\rangle_q |s\rangle_p
\]

where we have used that \( e^{i\hat{q} \hat{p}} |0\rangle_p = |s\rangle_p \).

- **2-pre) Measure:** Measure the input mode in the \( \hat{p} \) basis with outcome \( m \): this projects the second qumode into the state

\[
|\psi\rangle_{\text{out}} \propto \int dsf(s) e^{-im\hat{p}} |s\rangle_p = e^{-im\hat{p}} \int dsf(s) |s\rangle_p = X(m)F|\phi\rangle.
\]

The last equality is obtained since \( F|\phi\rangle = \int dsf(s) F|s\rangle_q = \int dsf(s) |s\rangle_p \). The effect of this circuit is to apply the identity (modulo a displacement and a rotation).

- **2) If I send as an input state the rotated state \( D_q |\phi\rangle = \int dsf(s) D_q |s\rangle_q \) where \( D_q = e^{if(\hat{q})} \) is an operator diagonal in the computational basis, then measuring \( \hat{p} \) in the first qumode projects the second mode into

\[
|\psi\rangle_{\text{out}} \propto X(m)F D_q |\phi\rangle.
\]

Since the \( C_Z \) gate commutes with \( D_q \), the same result is obtained with \( |\phi\rangle \) as an input state, and a rotation on the first mode after the \( C_Z \) as in the left panel of Fig.9.1. This is in turn equivalent to the situation in which no rotation is applied, but the first mode is measured in a rotated basis \( D_q^\dagger \hat{p} D_q = \hat{p}_{f(\hat{q})} \). The extra displacement \( X(m) \) depends on the outcome of the measurement on mode 1, and can be compensated by choosing the measurement basis of the following steps (thus introducing in general time-ordering).

- **3) Universality of single mode operations:** Repeat twice the previous protocol, for two different operators \( D_q^1 \) and \( D_q^2 \). The output state is:

\[
|\psi\rangle_{\text{out}} = X(m_2) F(D_q^2 X(m_1)) F D_q^1 |\phi\rangle
= X(m_2) F X(m_1) D_q^{m_2+1} F D_q^2 |\phi\rangle
= X(m_2) F X(m_1) F D_q^{2+1} D_q^1 |\phi\rangle
\]

(9.19)
where we have used the inequalities $X(-m)\hat{q}X(m) = \hat{q} + m$, $Z(-m)\hat{p}Z(m) = \hat{p} + m$, $F\dagger(-\hat{q})F = \hat{p}, F\dagger\hat{p}F = \hat{q}$. If instead of measuring the second mode on $\hat{p}_{f(\theta)}$ I would have measured it in the outcome-dependent basis $\hat{p}_{f(-\hat{q}-m_1)}$ I would have obtained as a result my deterministic desired output

$$|\psi\rangle_{out} = X(m_2) FX(m_1) F D_2 D_1 |\phi\rangle \quad (9.20)$$

(universal for single-mode operations if I repeat other times: I can obtain the desired transformation concatenating various $D_{\hat{q}}$ and $\hat{D}_{\hat{p}}$), a part from by-product operations which do not need to be corrected.

- **3)-4) Triviality of measurement adaptivity for gaussian unitaries** Let us focus on the building blocks of the universal set given above. For the gaussian operations:
  
  - $F$ is obtained at each step of the computation.
  - $D_{\hat{q}} = e^{i\hat{q}\theta}$ is obtained by measuring $\hat{p}_{s\hat{q}} = e^{-i\hat{q}\theta} \hat{p}_e e^{i\hat{q}\theta} = \hat{p} + s$ (measure $\hat{p}$ and add $s$ to the result). Note that $\hat{p}_{s\hat{q}+m} = \hat{p}_{s\hat{q}} = \hat{p} + s$ (no adaptation is required).
  - $D_{\hat{q}} = e^{i\hat{q}\frac{s^2}{2}}$ is obtained by measuring in the basis $\hat{p}_{s\hat{q}^2/2} = e^{-i\hat{q}\frac{s^2}{2}} \hat{p}_e e^{i\hat{q}\frac{s^2}{2}} = \hat{p} + s\hat{q} = g(\hat{q}\sin\theta + \hat{p}\cos\theta)$ with $g = \sqrt{1+s^2}$ and $\theta = \arctan s$. This is readily verified because the latter definition implies $\cos\theta = 1/\sqrt{1+s^2}$ and $\sin\theta = s/\sqrt{1+s^2}$. This corresponds to a rotated homodyne quadrature. Note that if I would have to adapt the basis I should measure according to $\hat{p}_{s(\hat{q}+m)^2/2} = \hat{p} + s\hat{q} + ms$. This can be achieved by measuring in the same basis as without adaptation (i.e. $\hat{p} + s\hat{q}$) and adding $ms$ to the result.

The adaptation required for these measurements is trivial and can be done after (as a post-processing). Hence gaussian operations can be implemented in any order or simultaneously ("parallelism").

A cubic phase gate would require instead

- $D_{\hat{q}} = e^{i\hat{q}\frac{\theta}{3}}$ is obtained by measuring in the basis $\hat{p}_{s\hat{q}^3/3} = e^{-i\hat{q}\frac{\theta}{3}} \hat{p}_e e^{i\hat{q}\frac{\theta}{3}} = \hat{p} + s\hat{q}^2 + 2ms\hat{q} + m^2s$. If I have to measure according to $\hat{p}_{s(\hat{q}+m)^3/3} = \hat{p} + s\hat{q}^2 + 2ms\hat{q} + m^2s$, the term $2ms\hat{q}$ requires a non-trivial adaptation of the measurement basis.

- **5) Cluster states as a resource:** Given the fact that the $C_Z$ gates commute with the measurements, in practice the state used as initial resource in the quantum computation protocol presented is a generalized cluster state in which some of the modes (the input modes), also linked to the other nodes of the cluster, are initialized to code the modes of the input state. However, one can think of taking an initial cluster state (e.g. a square cluster state) and "writing" in some of its nodes the modes of the physical input state (e.g. by $C_Z$ gates and measure of the input modes, or by teleportation [Ukai et al., 2010]). A state which allows this for each $U$ and each input state is said to be a universal resource. It has been demonstrated by Briegel that a square lattice graph (a cluster state) with unit weights is a universal resource for quantum computation. Depending on the specific kind of computation, other graphs than a square lattice could be more suitable for implementing the computation [Horodecki et al., 2006].

- **6) Two mode interactions:** A sequence of single mode operations can be implemented via following measurements on a linear cluster. To achieve full universality we have to add to the previous toolbox a two-mode interaction, e.g. the $C_Z$ gate. Such two-qubit gates, e.g., the CZ and CNOT gates, can be constructed in a two-dimensional cluster state where two input qubits are entangled with a few other qubits, in analogy to the case of qubit MBQC discussed in Sec.3.2. By a series of single-qubit measurements and rotations, we can end up with two of the other qubits representing the output state corresponding to the two-qubit gate having acted on the input state.
In conclusions, note that the procedure above is an idealization: in real life, squeezed states will always have finite energy, i.e. squeezing degree. As a result, the state output of the computation - even in the presence of ideal entangling gates and measurements - will always be affected by Gaussian noise, caused by the finite squeezing. How to avoid accumulation of this (and other types of) noise is the object of the following section.

9.3 GKP encoding and Error Correction

In classical informatics, when it comes to make sure that the errors that can occur during a computation can be corrected, it is convenient to resort to digitalized information, i.e. bits. For this reason combined with versatility, analog computers have been outperformed by digital computers in the 50s-60s, when the latter became sufficiently performant. Also note that from a computer science perspective, the definition of computational models based on real numbers is problematic and less studied\(^3\).

Analogously, with quantum information, if the goal is to achieve fault-tolerant quantum computation, we must resort to qubit-like quantum information even when using continuous-variable hardware. An example of qubit-like quantum information encoding in CV is based on the use of cat states, where the qubit-like information is encoded in codewords \(|0\rangle_L = |\alpha\rangle + | - \alpha\rangle\) and \(|1\rangle_L = |\alpha\rangle + | - i\alpha\rangle\), where we have omitted normalization constants. This encoding has allowed demonstrating a break-even point, in the sense that quantum information encoded in such cat states has been living longer than the one encoded in the corresponding qubit encoded in a two-level system (within a transmon architecture).

In Ref. [Gottesman et al., 2001], another way of encoding qubits in quantized harmonic oscillators was introduced by Gottesman, Kitaev and Preskill, yielding the GKP encoding. This encoding has been shown to allow for the correction of arbitrary type of noise, while cat codes are specialized to correct for single-photon losses. Essentially, and without attempting to be rigorous, this is because GKP codes allows one to correct for single-mode displacements, and any noise-map can be decomposed in single-mode displacements [Gottesman et al., 2001]. Also now that GKP state have been recently generated and encoded experimentally both in superconducting microwave cavities [Campagne-Ibarcq et al., 2019] and with trapped ions [Flühmann et al., 2018].

In what follows, we are going to introduce the GKP encoding and the corresponding error-correcting scheme in detail.

9.3.1 GKP encoding

GKP code-words

We start by recalling the basis of GKP encoding. The starting point relies on the definition of qubits as continuous wave-functions made of an infinite number of Dirac peaks [Gottesman et al., 2001]:

\[
|0_L\rangle = \sum_n |2n\sqrt{\pi}\rangle_q = \sum_n |n\sqrt{\pi}\rangle_p , \\
|1_L\rangle = \sum_n |(2n + 1)\sqrt{\pi}\rangle_q = \sum_n (-1)^n |n\sqrt{\pi}\rangle_p .
\]

(9.21)

Realistic logical qubit states are normalizable finitely squeezed states, rather than non-normalizable infinitely squeezed states. The Dirac peaks are hence replaced by a normalized Gaussian of width \(\Delta\), while the infinite

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\(^3\)If real computation were physically realizable, one could use it to solve NP-complete problems, and even \#P-complete problems, in polynomial time. Unlimited precision real numbers in the physical universe are prohibited by the holographic principle and the Bekenstein bound.
sum itself will become a Gaussian envelope function of width $\delta^{-1}$ (see Figure 9.2). Overall, the realistic states wavefunctions read:

$$\langle q | \tilde{0}_L \rangle = \sum_n \exp \left\{ -\left( \frac{(2n)^2 \pi \delta^2}{2} \right) \right\} \cdot \exp \left\{ -\left( \frac{(q - 2n \sqrt{\pi})^2}{2 \Delta^2} \right) \right\},$$

$$\langle q | \tilde{1}_L \rangle = \sum_n \exp \left\{ -\left( \frac{(2n+1)^2 \pi \delta^2}{2} \right) \right\} \cdot \exp \left\{ -\left( \frac{(q - (2n+1) \sqrt{\pi})^2}{2 \Delta^2} \right) \right\},$$

(9.22)

where we have introduced the noise distributions

$$G(u) = \frac{1}{\Delta \sqrt{2\pi}} e^{-\frac{u^2}{2\Delta^2}}, \quad F(v) = \frac{1}{\delta \sqrt{2\pi}} e^{-\frac{v^2}{2\delta^2}},$$

(9.23)

and $N_0$ and $N_1$ are normalization constants.

Figure 9.2: Wavefunction in position representation of GKP $|\tilde{0}_L\rangle$ state in continuous blue ($|\tilde{0}_L\rangle$ in dashed red) with $\delta = \Delta = 0.25$ from Equation Eq. (9.22).

Definition of CV universality (2)

A second definition of CV universality is based upon encodings, such as the GKP encoding: consider qubits encoded in CV hardware. In this case, universality is achieved when one can implement at least one of the universal gate sets for qubits quantum computation, that we introduced in Chapter 1, encoded in GKP.

On the (non-normalizable) states introduced in Eq.(9.21), Clifford operations correspond to the gates:

$$\bar{Z} = e^{i\sqrt{\pi} \hat{q}}, \quad \bar{C}_Z = e^{i\hat{q}_n \hat{q}}, \quad \bar{H} = F.$$  

(9.24)

These are all implemented by Gaussian CV operations, as introduced in Sec.9.1.1. To promote this set of operations to universality we need a non-Clifford gate. This requires instead a non-Gaussian operation:

$$\bar{T} = e^{\frac{i\pi}{4} \left[ \left( \frac{2\hat{q}}{\sqrt{\pi}} \right)^3 + \left( \frac{4}{\sqrt{\pi}} \right)^2 - \left( \frac{2\hat{q}}{\sqrt{\pi}} \right) \right]},$$

(9.25)
GKP encoding and fault-tolerance

In this Section, prepared with the contribution of Tom Douce, we prove that Continuous Variables MBQC with finite squeezing and an additional supply of GKP states yield fault tolerant quantum computation [Gottesman et al., 2001, Menicucci, 2014]. In Ref. [Gu et al., 2009] they showed how to implement standard quantum gates in CV MBQC, which would be sufficient for universal QC with GKP states [Gottesman et al., 2001], i.e. relying on a DV encoding embedded in a CV hardware. What remains to prove is that these gates can be performed fault-tolerantly, admitting use of GKP ancillary resource states. This was achieved in [Menicucci, 2014], where it is shown that the noise in the $\hat{p}$ quadrature of a GKP encoded quantum state can be replaced by the noise of the ancillary $|\tilde{0}\rangle_L$ state following the procedure shown in Fig. 9.3. Repeating this gadget after a Fourier transform allows for correction of the other quadrature, thereby enabling fault tolerance.

In order to explain this EC procedure, we follow a toy-model approach that has been developed by Glancy and Knill [Glancy and Knill, 2006]. This approach is based on a decomposition of the noise in several realizations of displacements, resulting in blurred ideal GKP states. Within this approach, we are going to show explicitly how GKP EC allows one to correct for displacements, by analyzing the output of the circuit in Fig. 9.4 with merely displaced perfect GKP states at the input, Sections 9.3.2 and 9.3.3. Since displacements form an operator basis, it follows that GKP states can correct any type of noise. Note that this works in principle for arbitrary noise, even when this is non gaussian.

This approach is exact in the infinite squeezing regime. For physical states, i.e. with finite squeezing there are subtleties. Some of these are discussed in a recent paper by Barbara Terhal’s group, that discusses the difference between error correction with coherent-enveloped GKP states versus blurred ideal GKP states. These subtle differences are not completely captured by the Glancy-Knill picture, but for pedagogical reasons we will omit further details of this discussion.

Figure 9.3: Procedure to correct for errors in the $\hat{p}$ quadrature. $|\tilde{0}\rangle_L$ is a noisy GKP state and $|\tilde{\psi}\rangle$ is a noisy GKP-encoded CV state. $X(m)$ is a displacement operator $e^{-im\hat{p}}$.

![Figure 9.3: Procedure to correct for errors in the $\hat{p}$ quadrature. $|\tilde{0}\rangle_L$ is a noisy GKP state and $|\tilde{\psi}\rangle$ is a noisy GKP-encoded CV state. $X(m)$ is a displacement operator $e^{-im\hat{p}}$.](image)

Figure 9.4: Modeling the noise in the protocol. $|\psi\rangle$ is a perfect – unphysical – GKP state and $|\tilde{\psi}\rangle$ is a perfect GKP-encoded CV state.

![Figure 9.4: Modeling the noise in the protocol. $|\psi\rangle$ is a perfect – unphysical – GKP state and $|\tilde{\psi}\rangle$ is a perfect GKP-encoded CV state.](image)

9.3.2 Single noise realization: intermediate measurement and threshold condition

Firstly, we compute the measurement result obtained at the output of the circuit in Fig. 9.4. The measurement is homodyne detection, corresponding to the observable $\hat{p} = \int p|s\rangle_p\langle s|$, featuring the projectors

$$\hat{P}_s = |s\rangle_p\langle s|$$

(9.26)
associated with the distinct measurement outcome \( s \), where \(|s\rangle_p \) are eigenstate of the \( \hat{p} \) operator. In the following, we are going to omit the subscript when this does not cause confusion. In the following and unless specified all integrals will run over the whole real axis.

More specifically, we’re interested in the observable \( \mathcal{I} \otimes \hat{p} \) where \( \mathcal{I} = \int dq_1 \ |q_1\rangle \langle q_1| \) is the identity operator, acting on mode 1 which is unmeasured. From the initial unphysical and perfect input states, a set of displacements in position and momentum are applied before the \( \hat{C}_Z \) gate and the measurement. Thus we want to work out the following quantity, in a sort of Heisenberg representation fashion:

\[
e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \hat{C}_Z^\dagger (\mathcal{I} \otimes \hat{p}) \hat{C}_Z e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

(9.27)

Step by step we have:

\[
\hat{C}_Z^\dagger (\mathcal{I} \otimes \hat{p}) \hat{C}_Z = \hat{C}_Z^\dagger \int dq_1 dp_2 dp_2^\prime |q_1, p_2\rangle \langle q_1, p_2^\prime| \hat{C}_Z
\]

(9.28)

Then:

\[
e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \hat{C}_Z^\dagger (\mathcal{I} \otimes \hat{p}) \hat{C}_Z e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

\[
= \int dq_1 dp_2 dq_1' dp_2' |q_1, p_2 \rangle \langle q_1, p_2^\prime| e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

\[
= \int dq_1 dp_2 |q_1 - u_1, p_2 - q_1 + v_2 \rangle \langle q_1 - u_1, p_2 - q_1 + v_2| e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

\[
= \int dq_1 dp_2 |q_1 - u_1, p_2 + u_1 - v_2 \rangle |q_1, p_2\rangle \langle q_1, p_2| e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

(9.29)

where in the last steps we have performed the changes of variables:

\[
q_1 - u_1 = q_1' + u_1
\]

\[
p_2 - q_1 + v_2 = p_2' + p_2 + q_1' + u_1 - v_2
\]

and have renamed \( q_1' \to q_1 \) and \( p_2' \to p_2 \). So what kind of measurement results are we to expect? To figure it out we should express Eq. (9.29) in a different manner, namely:

\[
e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \hat{C}_Z^\dagger (\mathcal{I} \otimes \hat{p}) \hat{C}_Z e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

\[
= \int dq_1 dp_2 (p_2 + u_1 - v_2) |q_1, p_2\rangle \langle q_1, p_2| e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}
\]

(9.30)

We recall that the quantum state that will be measured is \(|\psi_L, 0_L\rangle\) so a product of perfect GKP states. On these states, a quadrature projection can only give rise to integer multiples of \( \sqrt{\pi} \). So the sum \( q_1 + p_2 \) will be a multiple of \( \sqrt{\pi} \), say \( n \sqrt{\pi} \). Specifically we get

\[
s = n \sqrt{\pi} + u_1 - v_2.
\]

(9.31)

The outcome of the measurement is a value corresponding to the noise of the data qubit, blurred by the noise coming from the ancilla.

The error threshold is defined in relation with the following displacement \( X(-s \bmod \sqrt{\pi}) \). The modulo function has range \([-\sqrt{\pi}/2, \sqrt{\pi}/2]\), and the procedure succeeds if \( u_1 - v_2 \) is small. Otherwise a logical
error occurs and the displacement acts as Pauli-X gate in terms of the GKP encoding. Mathematically the constraint reads:

$$|u_1 - v_2| \leq \sqrt{\pi}/2,$$  \hspace{1cm} (9.32)

which translates into a constraint on the squeezing parameters [Menicucci, 2014].

9.3.3 Single noise realization: Output state of the GKP error-correcting gadget

We now compute the output state of the circuit Fig. 9.4. We show that the noise in the \( \hat{p} \) quadrature of the logical qubit is replaced by the one given by the ancilla \( \ket{\tilde{0}} \). Given Eq.(9.26) and standard quantum measurement theory [Nielsen and Chuang, 2011] after that the outcome \( s \) is obtained, the state is projected onto

$$|\psi_s\rangle \propto \tilde{P}_s |\psi\rangle_{12}$$  \hspace{1cm} (9.33)

where \( |\psi\rangle_{12} \) is the state of input and ancilla after displacement and \( \hat{C}_Z \) gate, i.e.

$$|\psi\rangle_{12} = \hat{C}_Z e^{-i u_1 \hat{p}_1} e^{-i v_1 \hat{q}_1} e^{-i u_2 \hat{p}_2} e^{-i v_2 \hat{q}_2} |\psi_L, 0_L\rangle,$$  \hspace{1cm} (9.34)

and where again the identity \( \mathcal{I} = \int dq_1 |q_1\rangle \langle q_1| \) is implicitly acted in mode 1, and the projector \( \tilde{P}_s \) defined in Eq.(9.26) acts on mode 2. We therefore now compute explicitly,

$$\tilde{P}_s |\psi\rangle_{12} = \int dq_1 |q_1\rangle \langle q_1| \otimes \langle s| \hat{C}_Z e^{-i u_1 \hat{p}_1} e^{-i v_1 \hat{q}_1} e^{-i u_2 \hat{p}_2} e^{-i v_2 \hat{q}_2} |\psi_L, 0_L\rangle \equiv |\Phi\rangle |s\rangle$$  \hspace{1cm} (9.35)

with \( |\Phi\rangle \) given by

$$|\Phi\rangle = \int dq_1 |q_1\rangle \langle q_1| \langle q_1, s| \hat{C}_Z e^{-i u_1 \hat{p}_1} e^{-i v_1 \hat{q}_1} e^{-i u_2 \hat{p}_2} e^{-i v_2 \hat{q}_2} |\psi_L, 0_L\rangle$$  \hspace{1cm} (9.36)

$$= \int dq_1 |q_1\rangle \langle q_1, s - q_1| e^{-i u_1 \hat{p}_1} e^{-i v_1 \hat{q}_1} e^{-i u_2 \hat{p}_2} e^{-i v_2 \hat{q}_2} |\psi_L, 0_L\rangle$$

$$= e^{i(v_1 u_1 - u_2 s)} \int dq_1 e^{-i(v_1 - u_2)q_1} |q_1\rangle \langle q_1 - u_1, s - q_1 + v_2|\psi_L, 0_L\rangle.$$

Now we make explicit use of the form of the variable \( s = u_1 + n\sqrt{\pi} - v_2 \), and we have

$$|\Phi\rangle = e^{i(v_1 u_1 - u_2 + u_1 + n\sqrt{\pi} - v_2)} \int dq_1 e^{-i(v_1 - u_2)q_1} |q_1\rangle \langle q_1 - u_1, n\sqrt{\pi} - q_1 + u_1|\psi_L, 0_L\rangle$$  \hspace{1cm} (9.37)

$$= e^{-i u_2 (n\sqrt{\pi} - v_2)} \int dq_1 e^{-i(v_1 - u_2)q_1} |q_1 + u_1\rangle \langle q_1, n\sqrt{\pi} - q_1|\psi_L, 0_L\rangle.$$

Now let’s go through the inner product:

$$\langle n\sqrt{\pi} - q_1|0_L\rangle = \sum_{l \in \mathbb{Z}} \langle n\sqrt{\pi} - q_1|l\sqrt{\pi}\rangle$$

$$= \sum_{l \in \mathbb{Z}} \delta(q_1 - (n - l)\sqrt{\pi})$$

$$= \sum_{l \in \mathbb{Z}} \delta(q_1 - l\sqrt{\pi}),$$  \hspace{1cm} (9.38)

so we have:

$$|\Phi\rangle = e^{-i u_2 (n\sqrt{\pi} - v_2)} \sum_{l \in \mathbb{Z}} \int dq_1 \delta(q_1 - l\sqrt{\pi}) e^{-i(v_1 - u_2)q_1} |q_1 + u_1\rangle \langle q_1|\psi_L\rangle$$

$$= e^{-i u_2 (n\sqrt{\pi} - v_2)} \sum_{l \in \mathbb{Z}} e^{-i(v_1 - u_2)l\sqrt{\pi}} |l\sqrt{\pi} + u_1\rangle \langle l\sqrt{\pi}|\psi_L\rangle.$$  \hspace{1cm} (9.39)
Since (the position wavefunction of) $|\psi_L\rangle$ is made of Dirac pikes on integer multiples of $\sqrt{\pi}$, the following result is straightforward: using that \( \sum_{l \in \mathbb{Z}} |l\sqrt{\pi}\rangle \langle l\sqrt{\pi}| \psi_L\rangle = |\psi_L\rangle \) we obtain the final state before the displacement:

$$|\Phi\rangle = e^{-iu_2(n\sqrt{\pi} - v_2)} e^{-iu_1\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle$$

(9.40)

and from Eq.(9.35)

$$\hat{P}_s |\psi_{12}\rangle = |\Phi\rangle |u_1 + n\sqrt{\pi} - v_2\rangle_p.$$  

(9.41)

The conditional state on mode 2 is obtained as:

$$\hat{\rho}_{k,\text{cond}1} = \text{Tr}_2[\hat{P}_s |\psi_{12}\rangle \langle \psi_{12}| \hat{P}_s]$$

(9.42)

$$= \int dp \delta(p - (u_1 + n\sqrt{\pi} - v_2)) |\Phi\rangle \langle \Phi| = |\Phi\rangle \langle \Phi|$$

(9.43)

corresponding to the pure state $|\Phi\rangle$.

Now we just have to deal with the remaining correction, i.e. the displacement by the actual measurement result obtained modulo $\sqrt{\pi}$, yielding $s \mod \sqrt{\pi} = (u_1 - v_2) \mod \sqrt{\pi}$

$$e^{is \mod \sqrt{\pi}}\hat{p}_1 |s\rangle \propto e^{is \mod \sqrt{\pi}}\hat{p}_1 |\Phi\rangle = e^{-iu_2(n\sqrt{\pi} - v_2)} e^{-iv_2\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle \equiv |\psi_L\rangle_{\text{out}},$$

(9.44)

which can also be expressed as

$$|\psi_L\rangle \longrightarrow e^{-iu_2(s - u_1 + n\sqrt{\pi})} e^{-iv_2\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle$$

(9.45)

with $p_0 = u_1 - v_2$. So we have the equation of the output state on a single realization of the noise. Like in [Glancy and Knill, 2006], we can see that the remaining $\hat{p}_1$ displacement is given by $v_2$ and is independent of the original noise $u_1$. For $\hat{q}_1$ though the noise from the ancilla $u_2$ has been added to the original value $v_1$. It appears more clearly if we write Eq. (9.45) as in from [Glancy and Knill, 2006]. Then the output state would read – for an intermediate measurement result $s$:

$$e^{-iu_2(s - u_1)} e^{-iv_2\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle$$

(9.46)

and we can see that the same discussion applies.

In Appendix B, we present the same calculation, but in the case where the homodyne detector has a finite resolution.
Appendix A

Quantization of the electromagnetic field in a cavity

This Appendix is taken from Ref. [Vikstål, 2018].

A.1 Quantizing the electromagnetic field

In this section we will consider the quantization of a single-mode electromagnetic field following Ref. [Gerry et al., 2005, Meystre and Sargent, 2007]. To quantize the electromagnetic field we will begin by considering a closed cavity of volume $V$ with mirrors of perfect reflection located at $z = 0$ and $z = L$. We imagine that we have a monochromatic, single-mode electromagnetic field that is assumed to be polarized along the $x$-direction.

![Figure A.1: Cavity with two perfectly reflecting mirrors located at $z = 0$ and $z = L$. The electric field is assumed to be polarized along the $x$-direction](image)

Figure A.1: Cavity with two perfectly reflecting mirrors located at $z = 0$ and $z = L$. The electric field is assumed to be polarized along the $x$-direction
In the absence of sources and charges the Maxwell equations read

\[ \nabla \times E = -\frac{\partial B}{\partial t}, \quad (A.1) \]
\[ \nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}, \quad (A.2) \]
\[ \nabla \cdot E = 0, \quad (A.3) \]
\[ \nabla \cdot B = 0. \quad (A.4) \]

Using Maxwell’s equations in the absence of sources and charges and the given boundary conditions, the electric field has the form

\[ E(z,t) = \left( \frac{2\omega^2 V \varepsilon_0}{k} \right)^{1/2} q(t) \sin(kz)e_x, \quad (A.5) \]

where \( V \) is the volume of the cavity, \( q(t) \) is a function of time and \( e_x \) is the unit-vector along the \( x \)-direction. From the boundary conditions the allowed frequencies are found to be

\[ \omega_n = \frac{c\pi n}{L}, \quad n = 1, 2, 3, \ldots \]

with \( k_n = \omega_n / c \) as the corresponding wave number. In Eq. (A.5) we have assumed a specific frequency \( \omega \), i.e. a specific standing wave which is called a mode of the field. We find the corresponding magnetic field by substituting Eq. (A.5) into (A.2),

\[ B(z,t) = \mu_0 \varepsilon_0 \frac{k}{\left( \frac{2\omega^2 V \varepsilon_0}{k} \right)^{1/2}} \dot{q}(t) \cos(kz)e_y. \quad (A.6) \]

We now identify \( q(t) \) as a canonical coordinate, and \( p(t) \equiv \dot{q}(t) \) as the momenta canonically conjugate to \( q(t) \). The energy stored in the field of the single-mode is

\[ \mathcal{H} = \frac{1}{2} \int_V \left( \frac{1}{\varepsilon_0} |E|^2 + \mu_0 |B|^2 \right) dV = \frac{1}{2} (p^2 + \omega^2 q^2). \]

We see that this is nothing but the energy of a harmonic oscillator with unit mass\(^1\). To quantize the electromagnetic field we promote \( q \) and \( p \) to operators

\[ q \rightarrow \hat{q}, \quad p \rightarrow \hat{p}, \]

and impose that they obey the canonical commutation relation

\[ [\hat{q}, \hat{p}] = i\hbar. \]

Thus the electric and magnetic field are also promoted to operators

\[ \hat{E}_x(z,t) = \left( \frac{2\omega^2 V \varepsilon_0}{k} \right)^{1/2} \hat{q} \sin(kz), \quad (A.7) \]
\[ \hat{B}_y(z,t) = \frac{\mu_0 \varepsilon_0}{k} \left( \frac{2\omega^2 V \varepsilon_0}{k} \right)^{1/2} \hat{p} \cos(kz). \quad (A.8) \]

The subscript \( x \) and \( y \) denotes the constituent components of the fields. The Hamiltonian now reads

\[ \hat{\mathcal{H}} = \frac{1}{2} (\hat{p}^2 + \omega^2 \hat{q}^2). \quad (A.9) \]

\(^1\)The amplitude of the electric field in Eq. Eq. (A.5) was cleverly chosen to yield the energy of a harmonic oscillator of unit mass.
Next we define the very useful non-Hermitian ladder operators

\begin{align}
\hat{a} &= (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p}), \\
\hat{a}^\dagger &= (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p}).
\end{align}

which obey the boson commutation relation

\[ [\hat{a}, \hat{a}^\dagger] = 1. \]

Inverting Eq. Eq. (A.10) and Eq. (A.11) and substituting it into Eq. (A.9) we get a Hamiltonian entirely written in terms of the ladder operators

\[ \hat{\mathcal{H}} = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}). \]

This Hamiltonian has the following energy eigenvalues

\[ E_n = \hbar\omega \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots \]

and the eigenstates of the Hamiltonian are known as Fock states, written in the Dirac notation as \( |n\rangle \), where \( n \) represents the number of quanta or photons in the single-mode field. The Fock states are eigenstates of the number operator \( \hat{n} = \hat{a}^\dagger \hat{a} \), satisfying

\[ \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle. \]

The vacuum state of the harmonic oscillator is defined by

\[ \hat{a} |0\rangle = 0. \tag{A.12} \]

Acting with the creation and annihilation operators on the Fock state yields

\begin{align}
\hat{a} |n\rangle &= \sqrt{n} |n - 1\rangle, \\
\hat{a}^\dagger |n\rangle &= \sqrt{n + 1} |n + 1\rangle. \tag{A.13}
\end{align}

Hence it is clear that the creation operator \( \hat{a}^\dagger \), creates a quanta of energy \( \hbar\omega \) and the annihilation operator \( \hat{a} \) destroys a quanta of energy \( \hbar\omega \) in the single-mode field. Any Fock state can be generated by acting on the vacuum state multiple times with the creation operator

\[ \frac{\hat{a}^\dagger^n}{\sqrt{n!}} |0\rangle = |n\rangle. \]

Furthermore the number states are orthogonal

\[ \langle m |n \rangle = \delta_{mn} \]

and form a complete set

\[ \sum_{n=0}^{\infty} |n\rangle \langle n| = 1. \]

So far we haven’t discussed the time-dependence of the operators. What we’ve done so far is assumed to hold at some time \( t \), for example \( t = 0 \). In the Schrödinger picture the states are time-dependent and the operators are time-independent. On the contrary, in the Heisenberg picture the operators are time-dependent and the
states are time-independent. In the Heisenberg picture the time evolution of the annihilation operator is given by

\[
\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}]
= \frac{i}{\hbar}[\hbar \omega (\hat{a}^{\dagger} \hat{a} + \frac{1}{2}), \hat{a}]
= i\omega (\hat{a}^{\dagger} \hat{a} - \hat{a} \hat{a}^{\dagger})
= i\omega [\hat{a}, \hat{a}^{\dagger}] \hat{a} = -i\omega \hat{a}.
\] (A.15)

Solving this equation yields

\[
\hat{a}(t) = \hat{a} e^{-i\omega t},
\] (A.16)

where \(\hat{a}(0) \equiv \hat{a}\). Taking the Hermitian adjoint of Eq. Eq. (A.16) we also find that

\[
\hat{a}^{\dagger}(t) = \hat{a}^{\dagger} e^{i\omega t}.
\]

After substituting Eq. Eq. (A.10) into Eq. (A.7) and Eq. Eq. (A.11) into Eq. (A.8), the electric and magnetic field with the inclusion of the time-dependence become, respectively

\[
\hat{E}_x(z, t) = \left(\frac{\hbar \omega}{V \varepsilon_0}\right)^{1/2} \left(\hat{a} e^{-i\omega t} + \hat{a}^{\dagger} e^{i\omega t}\right) \sin(kz),
\] (A.17)

\[
\hat{B}_y(z, t) = \frac{\mu_0 \varepsilon_0}{ik} \left(\frac{\hbar \omega^3}{V \varepsilon_0}\right)^{1/2} \left(\hat{a} e^{-i\omega t} - \hat{a}^{\dagger} e^{i\omega t}\right) \cos(kz).
\] (A.18)

### A.1.1 Quadrature operators

It is convenient to introduce the two dimensionless quantities

\[
\hat{X} = \sqrt{\frac{\omega}{2\hbar}} \hat{q} = \frac{1}{2} (\hat{a} + \hat{a}^{\dagger}),
\]

\[
\hat{X}_{\pi/2} = \frac{1}{\sqrt{2\hbar \omega}} \hat{p} = \frac{1}{2t} (\hat{a} - \hat{a}^{\dagger}),
\] (A.19)

which satisfy the commutation relation

\[\left[\hat{X}, \hat{X}_{\pi/2}\right] = \frac{i}{2}.
\]

From here it is easy to show that the electric field operator Eq. Eq. (A.17) can be written in terms of the dimensionless quantities \(\hat{X}\) and \(\hat{X}_{\pi/2}\) as

\[
\hat{E}_x(z, t) = 2 \left(\frac{\hbar \omega}{\varepsilon_0 V}\right)^{1/2} \left(\hat{X} \cos(\omega t) + \hat{X}_{\pi/2} \sin(\omega t)\right) \sin(kz).
\] (A.20)

This expression shows that \(\hat{X}\) and \(\hat{X}_{\pi/2}\) are associated with the electric field amplitude, where the second term is offset by \(\pi/2\) compared to the \(\cos(\omega t)\) term. The operators \(\hat{X}\) and \(\hat{X}_{\pi/2}\) are therefore known as the quadrature operators \(^2\).

The variance of an arbitrary operator is defined by

\[
\langle (\Delta \hat{A})^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2
\] (A.21)

\(^2\)Oscillating terms that are separated by \(90^{\circ} (\pi/2)\) are said to be in quadrature.
and can be interpreted as the uncertainty of an observable. The expectation value of the quadratures
\[ \langle \hat{X} \rangle = \frac{1}{2} \langle n | (\hat{a} + \hat{a}^\dagger) | n \rangle = 0, \]
\[ \langle \hat{X}_{\pi/2} \rangle = \frac{1}{2i} \langle n | (\hat{a} - \hat{a}^\dagger) | n \rangle = 0 \]
are evaluated to zero, which means that the expectation value of the electric field is also zero from Eq. Eq. (A.20). On the other hand the expectation value of the square is non-zero
\[ \langle \hat{X}^2 \rangle = \frac{1}{4} \langle n | (\hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) | n \rangle = \frac{1}{2} (1 + 2n), \]
\[ \langle \hat{X}_{\pi/2}^2 \rangle = \frac{1}{4} \langle n | (\hat{a}^2 + \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) | n \rangle = \frac{1}{2} (1 + 2n). \]
Thus it follows from Eq. Eq. (A.21) that the uncertainty in both quadratures are equal, and when \( n = 0 \) (corresponding to the vacuum state), the uncertainty is minimum
\[ \langle (\Delta \hat{X})^2 \rangle_{\text{vac}} = \frac{1}{4} = \langle (\Delta \hat{X}_{\pi/2})^2 \rangle_{\text{vac}}. \] (A.22)
This is known as the vacuum fluctuations. Moreover, in quantum optics the quadrature operators \( \hat{X} \) and \( \hat{X}_{\pi/2} \) are commonly labelled \( \hat{q} \) and \( \hat{p} \), since they are more used than the position and momentum operators.

### A.2 Coherent states

In quantum optics, the coherent states are the states with most resemblance to classical states, in the sense that they give rise to expectation values that look like the classical electric field.

To construct a state with close resemblance to the classical electromagnetic field, one can observe that by replacing \( \hat{a} \) and \( \hat{a}^\dagger \) with a complex variable in Eq. Eq. (A.17) and Eq. (A.18) it would produce a “classical field”, i.e. a field that oscillates. In order to achieve this, one can define a coherent state to be an eigenstate to the annihilation operator
\[ \hat{a} | \alpha \rangle = \alpha | \alpha \rangle. \] (A.23)
Because \( \hat{a} \) is non-Hermitian \( \alpha \) is usually complex. For the creation operator \( \hat{a}^\dagger \) we have for obvious reasons
\[ \langle \alpha | \hat{a}^\dagger = \alpha^\ast \langle \alpha \rangle. \]
Since the Fock states form a complete set, we will use them to express \( \alpha \)
\[ | \alpha \rangle = \sum_{n=0}^{\infty} c_n | n \rangle, \] (A.24)
where \( c_n = \langle n | \alpha \rangle \) denotes a complex number which is to be determined. Inserting Eq. Eq. (A.24) into Eq. (A.23) and using Eq. Eq. (A.12) and Eq. (A.13) we obtain
\[ \sum_{n=1}^{\infty} c_n \sqrt{n} | n - 1 \rangle = \sum_{n=0}^{\infty} c_n \alpha | n \rangle. \]
Since the Fock-states form an orthogonal basis, we can multiply with an arbitrary state \( | m \rangle \) from left and use the orthogonality condition \( \langle m | n \rangle = \delta_{mn} \) to obtain
\[ c_{m+1} \sqrt{m+1} = \alpha c_m. \]
By the substitution \( m \to n - 1 \)
\[
c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} c_{n-2} = \ldots = \frac{\alpha^n}{\sqrt{n!}} c_0,
\]
we obtain a recursion formula. Hence Eq. Eq. (A.24) can be expressed as
\[
|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.
\]

We use the normalization condition to determine the coefficient \(|c_0|^2\),
\[
\langle \alpha \rangle = |c_0|^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^m \alpha^n}{\sqrt{m!} \sqrt{n!}} \langle m | n \rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^2}{n!} = |c_0|^2 e^{|\alpha|^2}.
\]

Therefore \(|c_0| = e^{-|\alpha|^2/2}\) and our final expression for the coherent state \(|\alpha\rangle\) expressed in terms of Fock states is
\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\]
which is a superposition of infinite many number of Fock states! Furthermore calculating the expectation value of the number operator \(\hat{n}\)
\[
\bar{n} = \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2,
\]
we see that \(|\alpha|^2\) is related to the mean number of photons in the field. Using this we can compute the probability of finding \(n\) photons in the field
\[
|\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^\bar{n} \frac{\bar{n}^n}{n!},
\]
which we recognize as a Poisson distribution with a mean of \(\bar{n}\). This distribution arises when the probability that an event occurs is independent of earlier events.

Let us now consider the expectation value of the electric field given by Eq. Eq. (A.17)
\[
\langle \hat{E}_x(z, t) \rangle_\alpha = \langle \alpha | \hat{E}_x(z, t) | \alpha \rangle = \left( \frac{\hbar \omega}{V \varepsilon_0} \right)^{1/2} \langle \alpha | (\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) | \alpha \rangle \sin(kz)
\]
\[
= \left( \frac{\hbar \omega}{V \varepsilon_0} \right)^{1/2} \langle \alpha | e^{-i\omega t} + \alpha^* e^{i\omega t} | \alpha \rangle \sin(kz).
\]

Writing \(\alpha\) in polar coordinates \(\alpha = |\alpha| e^{i\phi}\) we get
\[
\langle \hat{E}_x(z, t) \rangle_\alpha = \left( \frac{\hbar \omega}{V \varepsilon_0} \right)^{1/2} 2 |\alpha| \cos(\omega t - \phi) \sin(kz),
\]
and we see that the field oscillates very much like the classical electric field. Likewise, for the quadrature operator Eq. Eq. (A.19), we have
\[
\langle \hat{X} \rangle_\alpha = \frac{1}{2} \langle \alpha | (\hat{a} + \hat{a}^\dagger) | \alpha \rangle = \frac{1}{2} (\alpha + \alpha^*) = \text{Re} \alpha = |\alpha| \cos(\phi)
\]
and similarly \(\langle \hat{X}_{\pi/2} \rangle_\alpha = \text{Im} \alpha = |\alpha| \sin(\phi)\), so the mean of the quadratures are related to the real and imaginary part of \(\alpha\). It can be easily verified that the quantum uncertainty for both quadratures are
\[
\langle (\Delta \hat{X})^2 \rangle_\alpha = \frac{1}{4} = \langle (\Delta \hat{X}_{\pi/2})^2 \rangle_\alpha,
\]
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Figure A.2: Phase-space diagram of a coherent state $|\alpha\rangle$. The displacement from the origin is equal to $|\alpha|$ and the angle $\phi$ is the phase, measured from the $X$-axis. The quantum uncertainty is displayed as a grey circle with a diameter of $1/2$.

which shows that they still exhibit the fluctuations of the vacuum, cf. Eq. Eq. (A.22).

A neat way to illustrate a coherent state is in phase space. More of it will be explained in section A.3, but for now it is sufficient to think of phase space as a mathematical abstract space, where the state of a harmonic oscillator is represented in terms of its quadratures. Since position $\hat{X}$ and momentum $\hat{X}_{\pi/2}$ are non commutative operators in quantum physics, the state can’t be represented as a point in phase space as it would in classical physics, because both position and momentum aren’t allowed to have precise values. In Fig. A.2 the phase space diagram of a coherent state $|\alpha\rangle$ is illustrated. The coherent state can be viewed as a displacement of the vacuum state with a distance $|\alpha|$ from the origin and an angle $\phi$ measured from the $\hat{X}$-axis. The grey circle area represents the uncertainty of the coherent state and has a constant diameter of $1/2$.

Non-orthogonality

Coherent states are known to be quasi-orthogonal. For example, consider the scalar product $\langle \beta | \alpha \rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ are to different coherent states

$$
\langle \beta | \alpha \rangle = e^{-|\beta|^2/2}e^{-|\alpha|^2/2} \sum_{m} \sum_{n} \frac{(\beta^m)^* \alpha^n}{\sqrt{m!}\sqrt{n!}} \langle m | n \rangle
$$

$$
= e^{-(|\beta|^2+|\alpha|^2)/2} \sum_{n} \frac{(|\beta|^n)^* \alpha^n}{n!}
$$

$$
= e^{-(|\beta|^2+|\alpha|^2-2|\beta|^*\alpha)/2}
$$

$$
= e^{-|\alpha-\beta|^2/2}e^{(\alpha^*\beta - \beta^*\alpha)/2},
$$

taking the modulus square we get

$$
|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha-\beta|^2}.
$$

(A.25)

From Eq. Eq. (A.25) it is evident that two coherent states are non-orthogonal. Only when $|\alpha - \beta|^2$ is large, so that $|\langle \beta | \alpha \rangle|^2 \sim 0$, they become quasi-orthogonal.
A.3 Phase space representation

In quantum mechanics a system can be fully described by its density operator $\hat{\rho}$, however the density operator can be a rather abstract object and it can be hard to read off its properties. Therefore we employ the phase space representation, that is based on the concept of the Wigner function, which provides useful means of visualizing a quantum state.

Because position and momenta are so closely related to the quadrature operators, the phase space representation is a useful tool for studying the behaviour of continuous variable systems. This can be done by changing from the representation of position and momentum to the representation of quadrature operators.

A.3.1 Wigner function

The Wigner function is defined by

$$W(q,p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \langle q + \frac{1}{2}x | \hat{\rho} | q - \frac{1}{2}x \rangle e^{ixp/\hbar} dx$$

where $x, q$ and $p$ are now interpreted as quadratures, and $\hat{\rho}$ is the density operator for a quantum system. $W(q,p)$ is known as a quasi-probability distribution since it can take on negative values. Even though the Wigner function is not a “real” probability distribution it can still be associated to one, for example calculating the probability distribution (also referred to as the marginal distribution) over $p$

$$Pr(q) = \int_{-\infty}^{\infty} W(q,p) dp = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \psi^*(q - \frac{1}{2}x)\psi(q + \frac{1}{2}x)e^{ixp/\hbar}$$

$$= \int_{-\infty}^{\infty} dx \psi^*(q - \frac{1}{2}x)\psi(q + \frac{1}{2}x)\delta(x)$$

$$= \psi^*(q)\psi(q) = |\psi(q)|^2$$

returns the probability density of $q$. Similarly, one can show that

$$Pr(p) = \int_{-\infty}^{\infty} W(q,p) dq = |\tilde{\psi}(p)|^2$$

is the probability density of $p$, where $\tilde{\psi}(p)$ is the wave function in $p$-representation.

As mentioned in the introduction, negativity of the Wigner function is related to non classical simulatibility according to a theorem proven by Eisert and Mari [Mari and Eisert, 2012]. In their work they show that a negative Wigner function can be identified as a quantum computational resource (similar to superposition), and that a quantum circuit where the initial state and all the following quantum operations are represented by positive Wigner functions can be classically efficiently simulated.

Note that both the vacuum and the coherent state is shaped like a Gaussian which does not display any negativity, while the Wigner function for the single-photon Fock state shows clear indication of negativity.
Appendix B

GKP Error-correction gadget with finite resolution

In this Appendix, we re-derive the calculation of Sec. 9.3.2, but taking into account that the homodyne detector might have a finite resolution.

B.0.1 Single noise realization: intermediate measurement and threshold condition with finite resolution detectors

Firstly, we compute the measurement result obtained with finite resolution homodyne detection. As we have defined in the main text, the associated observable is a modified $\hat{p}$ operator, associated with finitely resolved homodyne detectors, that we report here for convenience:

$$\hat{p}^\eta = \sum_{k=-\infty}^{\infty} p_k \int_{-\infty}^{\infty} dp \chi^\eta_k(p) |p\rangle\langle p| \equiv \sum_{k=-\infty}^{\infty} p_k \hat{P}_k$$ (B.1)

with $\chi^\eta_k(p) = 1$ for $p \in [p_k - \eta, p_k + \eta]$, $p_k = 2\eta k$, and where we have defined the projectors

$$\hat{P}_k = \int_{p_k - \eta}^{p_k + \eta} dp |p\rangle\langle p|$$ (B.2)

associated with the distinct measurement outcome. In the following and unless specified all integrals will run over the whole real axis.

More specifically, we’re interested in the observable $\mathcal{I} \otimes \hat{p}^\eta$ where $\mathcal{I} = \int dq \ |q_1\rangle\langle q_1|$ is the identity operator, acting on mode 1 which is unmeasured. From the initial unphysical and perfect input states, a set of displacements in position and momentum are applied before the $\hat{C}_Z$ gate and the measurement. Thus we want to work out the following quantity, in a sort of Heisenberg representation fashion:

$$e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \hat{C}_Z^\dagger (\mathcal{I} \otimes \hat{p}^\eta) \hat{C}_Z e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2}$$ (B.3)

Step by step we have:

$$\hat{C}_Z^\dagger (\mathcal{I} \otimes \hat{p}^\eta) \hat{C}_Z = \hat{C}_Z^\dagger \sum_k p_k \int dq_1 dp_2 \chi^\eta_k(p_2) |q_1, p_2\rangle \langle q_1, p_2| \hat{C}_Z$$

$$= \sum_k p_k \int dq_1 dp_2 \chi^\eta_k(p_2) |q_1, p_2 - q_1\rangle \langle q_1, p_2 - q_1|$$ (B.4)
Then:

\[ e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \hat{C}_Z^\dagger (I \otimes \hat{p}_\eta) \hat{C}_Z e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2} \]

\[ = e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \sum_k p_k \int dq_1 dp_2 \chi_k^n(p_2) |q_1 - p_1 - q_1 - v_2, p_2 - q_1 + v_2| \]

\[ = \sum_k p_k \int dq_1 dp_2 \chi_k^n(p_2) |q_1, p_2 - q_1 - u_2, u_1 - v_2| \]

Now to check the validity of the above equation let’s consider the limit \( \eta \rightarrow 0 \) corresponding to infinite precision. Based on \( \sum_k p_k \int dp \chi_k^n(p) |f(p)| \langle f(p) \rangle \rightarrow \int dp f^{-1}(p) |p| \langle p \rangle \) we get, as it should:

\[ \int dq_1 dp_2 (q_1 + p_2 + u_1 - v_2) |q_1, p_2 \rangle \langle q_1, p_2 | = \hat{q}_1 + \hat{p}_2 + u_1 - v_2. \]  

(B.6)

So what kind of measurement results are we to expect? To figure it out we should express Eq. (B.5) in a different manner, namely:

\[ e^{iv_2 \hat{q}_2} e^{iu_2 \hat{p}_2} e^{iv_1 \hat{q}_1} e^{iu_1 \hat{p}_1} \hat{C}_Z^\dagger (I \otimes \hat{p}_\eta) \hat{C}_Z e^{-iu_1 \hat{p}_1} e^{-iv_1 \hat{q}_1} e^{-iu_2 \hat{p}_2} e^{-iv_2 \hat{q}_2} \]

\[ = \sum_k p_k \int dq_1 dp_2 \chi_k^n(p_2 + q_1 - u_2, v_2) |q_1, p_2 \rangle \langle q_1, p_2 | \]  

(B.7)

We recall that the quantum state that will be measured is \( |\psi_L, 0_L \rangle \) so a product of perfect GKP states. On these states, a quadrature projection can only give rise to integer multiples of \( \sqrt{\pi} \). So the sum \( q_1 + p_2 \) will be a multiple of \( \sqrt{\pi} \), say \( n \sqrt{\pi} \). Then because of the finite resolution, the measurement outcome will be the one of the \( p_k \)'s within range \( \eta \) of \( n \sqrt{\pi} + u_1 - v_2 \). Specifically we get

\[ p_k = n \sqrt{\pi} + u_1 - v_2 + \lambda^* = 2\eta k \]  

(B.8)

with \( -\eta < \lambda^* < \eta \), where by virtue of Eq.(B.1) we have also explicited that the resulting outcome corresponds to one of the pixels of the finitely resolved homodyne detector. In other words we get a somewhat similar result in this finite resolution scenario with respect to the perfect case. Indeed infinite resolution, that is \( \eta = 0 \), would consistently yield \( n \sqrt{\pi} + u_1 - v_2 \). The outcome of the measurement is a value corresponding to the noise of the data qubit, blurred by the noise coming from the ancilla and all this is known up to \( \eta \).

The error threshold is defined in relation with the following displacement \( X(-s \mod \sqrt{\pi}) \). The modulo function has range \([-\sqrt{\pi}/2, \sqrt{\pi}/2]\), and the procedure succeeds if \( u_1 - v_2 \) is small. Otherwise a logical error occurs and the displacement acts as Pauli-\( X \) gate in terms of the GKP encoding. Mathematically the constraint reads: \(|u_1 - v_2| \leq \sqrt{\pi}/2\) which translates into a constraint on the squeezing parameters. In our case and to prevent logical errors, we need \(|u_1 - v_2 + \lambda^*| \leq \sqrt{\pi}/2\,\), and given the range of possible \( \lambda^* \)'s, in the end the error threshold reads:

\[ |u_1 - v_2| \leq \sqrt{\pi}/2 - \eta. \]  

(B.9)

This inequality is the generalization of the error threshold condition of [Menicucci, 2014] to the case of finite resolution in the homodyne detection.

**B.0.2 Single noise realization: Output state of the GKP error-correcting gadget**

We now compute the output state of the circuit Fig. 9.4 for finite resolution detection. We show that the noise in the \( \hat{p} \) quadrature of the logical qubit is replaced by the one given by the ancilla \( |\hat{0}_L \rangle \). We have seen
that with finite resolution the outcome of the measurement is of the form given Eq. (B.8). Given Eq. (B.1) standard quantum measurement theory [Nielsen and Chuang, 2011] after that the outcome \( p_k \) is obtained, the state is projected onto

\[
|\psi_k\rangle = \frac{\hat{P}_k|\psi\rangle_{12}}{\sqrt{\text{Prob}[p_k]}}
\]

with probability

\[
\text{Prob}[p_k] = \langle \psi |_{12} \hat{P}_k |\psi\rangle_{12}
\]

where \( |\psi\rangle_{12} \) is the state of input and ancilla after displacement and \( \hat{C}_Z \) gate, i.e.

\[
|\psi\rangle_{12} = \hat{C}_Ze^{-iu_1\hat{p}_1}e^{i\nu_1\hat{q}_1}e^{-iu_2\hat{p}_2}e^{-i\nu_2\hat{q}_2}|\psi_L,0_L\rangle,
\]

and where again the identity \( I = \int dq_1 |q_1\rangle \langle q_1| \) is implicitly acted in mode 1, and the projector \( \hat{P}_k \) defined in Eq. (B.1) acts on mode 2. We therefore now compute explicitly (relabeling \( p_\lambda \) the integration variable of the projector defined in Eq. (B.1))

\[
\hat{P}_k |\psi\rangle_{12} = \int dq_1 |q_1\rangle \langle q_1| \otimes \int_{p_k-\eta}^{p_k+\eta} dp_\lambda |p_\lambda\rangle \langle p_\lambda| \hat{C}_Ze^{-iu_1\hat{p}_1}e^{i\nu_1\hat{q}_1}e^{-iu_2\hat{p}_2}e^{-i\nu_2\hat{q}_2}|\psi_L,0_L\rangle \equiv \int_{p_k-\eta}^{p_k+\eta} dp_\lambda |\text{temp}_\lambda\rangle |p_\lambda\rangle
\]

with \( |\text{temp}_\lambda\rangle \) given by

\[
|\text{temp}_\lambda\rangle = \int dq_1 |q_1\rangle \langle q_1|p_\lambda\rangle \hat{C}_Ze^{-iu_1\hat{p}_1}e^{i\nu_1\hat{q}_1}e^{-iu_2\hat{p}_2}e^{-i\nu_2\hat{q}_2}|\psi_L,0_L\rangle
\]

(14)

\[
= \int dq_1 |q_1\rangle \langle q_1|p_\lambda - q_1|e^{-iu_1\hat{p}_1}e^{i\nu_1\hat{q}_1}e^{-iu_2\hat{p}_2}e^{-i\nu_2\hat{q}_2}|\psi_L,0_L\rangle
\]

\[
= e^{i(v_1u_1-u_2p_\lambda)} \int dq_1 e^{-i(v_1-u_2)q_1} |q_1\rangle \langle q_1 - u_1, p_\lambda - q_1 + v_2|\psi_L,0_L\rangle.
\]

Now we make explicit use of the form of the variable \( p_\lambda \). Due to the definition in Eq. (B.1) it can take values \( p_k + r \) with \( -\eta \leq r \leq \eta \) and \( p_k \) given by Eq. (B.8). Therefore we replace \( p_\lambda = u_1 + n\sqrt{\pi} - v_2 + \lambda \), where we have set \( \lambda^* + r \equiv \lambda \), that can hence take values \( -2\eta \leq \lambda \leq 2\eta \), and we have

\[
|\text{temp}_\lambda\rangle = e^{i(v_1u_1-u_2(u_1+n\sqrt{\pi}-v_2+\lambda))} \int dq_1 e^{-i(v_1-u_2)q_1} |q_1\rangle \langle q_1 - u_1, n\sqrt{\pi} - q_1 + u_1 + \lambda|\psi_L,0_L\rangle
\]

(15)

\[
= e^{-iu_2(n\sqrt{\pi}-v_2)}e^{-i\lambda v_1} \int dq_1 e^{-i(v_1-u_2)q_1} |q_1 + u_1 + \lambda\rangle \langle q_1 + \lambda, n\sqrt{\pi} - q_1|\psi_L,0_L\rangle
\]

Now let’s go through the inner product:

\[
\langle n\sqrt{\pi} - q_1|0_L\rangle = \sum_{l\in\mathbb{Z}} \langle n\sqrt{\pi} - q_1|l\sqrt{\pi}\rangle
\]

\[
= \sum_{l\in\mathbb{Z}} \delta(q_1 - (n - l)\sqrt{\pi})
\]

\[
= \sum_{l\in\mathbb{Z}} \delta(q_1 - l\sqrt{\pi})
\]

(16)

So we have:

\[
|\text{temp}_\lambda\rangle = e^{-iu_2(n\sqrt{\pi}-v_2)}e^{-i\lambda v_1} \sum_{l\in\mathbb{Z}} \int dq_1 \delta(q_1 - l\sqrt{\pi})e^{-i(v_1-u_2)q_1} |q_1 + u_1 + \lambda\rangle \langle q_1 + \lambda|\psi_L\rangle
\]

\[
= e^{-iu_2(n\sqrt{\pi}-v_2)}e^{-i\lambda v_1} \sum_{l\in\mathbb{Z}} e^{-i(v_1-u_2)l\sqrt{\pi}} |l\sqrt{\pi} + u_1 + \lambda\rangle \langle l\sqrt{\pi} + \lambda|\psi_L\rangle
\]

(17)
The extremal values for $\lambda$ can be as large as $\pm 2\eta$, which is assumed to be much lower than $\sqrt{\pi}$. Since (the position wavefunction of) $|\psi_L\rangle$ is made of Dirac pikes on integer multiples of $\sqrt{\pi}$, the following result is straightforward:

$$|\text{temp}_\lambda\rangle = 0 \Leftrightarrow \lambda \neq 0$$

So only $\lambda = 0$ contributes to the projection for the output state. Thus in some sense we are back to the infinite precision homodyne detection. We stress that this result stems from the modeling of noisy GKP states as displaced perfect ones. Therefore, replacing $\lambda = 0$ and using that $\sum_{l \in \mathbb{Z}} |l\sqrt{\pi}\rangle \langle l\sqrt{\pi}|\psi_L\rangle = |\psi_L\rangle$ we obtain the final state before the displacement:

$$|\text{temp}_\lambda=0\rangle = e^{-iu_2(n\sqrt{\pi} - v_2)} e^{-iu_1\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle \quad (B.24)$$

and from Eq. (B.13)

$$\hat{P}_k |\psi_{12}\rangle = \int_{-\infty}^{\infty} dp \chi_k^n(p,\lambda) |\text{temp}_\lambda=0\rangle |p,\lambda=0\rangle \delta(p,\lambda - p,\lambda = 0) = |\text{temp}_\lambda=0\rangle |u_1 + n\sqrt{\pi} - v_2\rangle_p \quad (B.19)$$

where we have used $\chi_k^n(p,\lambda=0) = 1$.

The conditional state on mode 2 is obtained as:

$$\rho_{k,\text{cond}1} = \text{Tr}_2[\hat{P}_k |\psi_{12}\rangle \langle \psi_{12}| \hat{P}_k] = \int dp \delta(p - (u_1 + n\sqrt{\pi} - v_2)) |\text{temp}_\lambda=0\rangle \langle \text{temp}_\lambda=0| = |\text{temp}_\lambda=0\rangle \langle \text{temp}_\lambda=0| \quad (B.20)$$

corresponding to the pure state $|\text{temp}_\lambda=0\rangle$.

Now we just have to deal with the remaining imperfect correction, i.e. the displacement by the actual measurement result obtained modulo $\sqrt{\pi}$, yielding $p_k \bmod \sqrt{\pi} = u_1 - v_2 + \lambda^* = (2kn) \bmod \sqrt{\pi}$

$$e^{ip_k \bmod \sqrt{\pi}} \hat{p}_1 |\psi_k\rangle \propto e^{ip_k \bmod \sqrt{\pi}} \hat{p}_1 |\text{temp}_\lambda=0\rangle = e^{-iu_2(n\sqrt{\pi} - v_2)} e^{-i(v_2 - \lambda^*)\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle \equiv |\psi_{\text{out}}\rangle \quad (B.21)$$

which can also be expressed as

$$|\psi_L\rangle \rightarrow e^{-iu_2(p_0 - u_1 + n\sqrt{\pi})} e^{i\lambda^*\hat{p}_1} e^{-iv_2\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle \quad (B.22)$$

with $p_0 = u_1 - v_2$. So we have the equation of the output state on a single realization of the noise. Like in [Glancy and Knill, 2006], we can see that the remaining $\hat{p}_1$ displacement is given by $v_2$ and is independent of the original noise $u_1$. For $\hat{q}_1$ though the noise from the ancilla $u_2$ has been added to the original value $v_1$. It appears more clearly if we write Eq. (B.23) in the infinite resolution case, recovering exactly the result from [Glancy and Knill, 2006]. Then the output state would read – for an intermediate measurement result $s$:

$$e^{-iu_2(s - u_1)} e^{-iv_2\hat{p}_1} e^{-i(v_1 - u_2)\hat{q}_1} |\psi_L\rangle \quad (B.24)$$

and we can see that the same discussion applies.

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Bibliography


