OPTIMAL TRANSPORT: WARMING UP BY PUSHING FORWARD (5/9-13)

In the following we assume given a map $T : X \to Y$ from a space X to another space Y.

(1) Let μ be a "pre-measure" (this is non-standard notation), i.e. real-valued function μ on the space of all subset of X which is *additive,* i.e. $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ for any two disjoint subsets E_1 and E_2. Define the *push-forward of μ under T*, denoted by $\nu := T_*\mu$, by

$$(T_*\mu)(F) := \mu(T^{-1}(F))$$

if F is a subset of Y. Check that ν is also a pre-measure.

(2) It may also be tempting to, given a pre-measure ν on Y define the "pull-back" $T_*\nu$ by

$$(T^*\nu)(E) := \mu(T(E))$$

But the pull-back operator does not preserve additivity, i.e. even if ν is additive $T_*\nu$ may not be additive. Why?

(3) Can you give a condition on T ensuring that $T^*\nu$ preserves additivity? First give a condition which works for any ν and then give a weaker condition which depends on ν.

(4) Give a function g on Y recall that the pull-back T^*g of g under T is the function f on X defined by

$$f(x) := g(T(x))$$

Check that (under suitable regularity assumptions) $\int_F g d(T_*\mu) = \int_{T'^1(F)} T^* g d\mu$.

(5) Conversely, show that if ν is a measure on Y such that

$$\int_F d\nu g = \int_{T'^1(F)} d\mu T^* g$$

for any g, then $\nu = T_*\mu$

(6) Let $T : X \to Y$ be a map transporting a probability measure μ on X to the measure ν on Y, i.e. $\nu = T_*\mu$. Denote by $I \times T$ the map $X \to X \times Y$ defined by

$$(I \times T)(x) = (x, T(x))$$

Check that $\gamma_T := \mu_*(I \times T)$ is indeed a *transport plan* between μ and ν, i.e. a probability measure on $X \times X$ whose first and second marginals coincide with μ and ν, respectively. Also check that γ_T is supported on the graph of T i.e. on the set of all (x, y) such that $y = T(x)$.

(7) Conversly, show that if γ is a transport plan between two probability measures μ and ν, such that the support of γ is contained in the graph of a map T, then $\gamma = \gamma_T$.

(8) Show that if γ is a probability measure on $X \times Y$ then it has marginals μ and ν iff

$$\int_{X \times Y} (\phi(x) + \psi(y)) d\gamma = \int_X \phi d\mu + \int_Y \psi d\nu$$

for any pair of (measurable and integrable) functions ϕ and ψ on X and Y, respectively.

(9) Let μ be a probability measure on X and y_0 a point in Y. Setting $\nu = \delta_{y_0}$, i.e. the Dirac mass at y_0, show that there is a unique transport plan γ from μ to ν. More precisely, show that $\gamma = \gamma_T$ where T is the map $T(x) = y_0$ for any $x \in X$.

(10) Given a smooth strictly convex function ϕ on $\mathbb{R}^n$: (i.e. the Hessian matrix $\frac{\partial^2 \phi}{\partial x_i \partial x_j}$ is positive definite) define the map $T : \mathbb{R}^n :\to \mathbb{R}^n :$ by $T(x) = \nabla \phi(x)$ (the gradient of ϕ at x). Check that if $\mu = f dx$ and $\nu = g dy$ are measures on $\mathbb{R}^n$: with smooth densities f and g, then

$$T_* \mu = \nu$$

if and only if

$$\det(\frac{\partial^2 \phi}{\partial x_i \partial x_j}) g(\nabla \phi) = f(x),$$