School on
Stochastic Partial Differential Equations
Part 3

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Nemytskij operator

$f \in \mathcal{C}_{\mathrm{b}}^2(\mathbf{R}) = \mathcal{C}_{\mathrm{b}}^2(\mathbf{R},\mathbf{R})$ means that f has two continous derivatives, the first and second derivatives are bounded but the zeroth derivative need not be bounded, but it follows that it grows at most linearly: (seminorm) $|f|_{\mathcal{C}_{\mathrm{b}}^1} := \|f'\|_{L_\infty(\mathbf{R})} \le C$ implies
$|f(s)| = |f(0) + f'(\xi)s| \le |f(0)| + |f|_{\mathcal{C}_{\mathrm{b}}^1}|s| \le C(1+|s|)$.

$$f \in \mathcal{C}_{\mathrm{b}}^2(\mathbf{R}), \quad F\colon H \to H, \quad F(u)(x) = f(u(x))$$
$$\|F(u)\|_H = \|f(u)\|_{L_2} \le C\|1+|u|\|_{L_2} \le C(1+\|u\|_H)$$

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$$\|F'(u)v\|_H = \|f'(u)v\|_{L_2} \le \|f'(u)\|_{L_\infty}\|v\|_{L_2} \le |f|_{\mathcal{C}^1_{\mathrm{b}}}\|v\|_H$$
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So $F \in \mathcal{C}^1_{\mathrm{b}}(H)$ and hence globally Lipschitz $H \to H$. Further: ($\mathcal{L}^{(2)}(H)$ means bounded bilinear operators $H \to H$)

$$F''(u) \in \mathcal{L}^{(2)}(H), \quad (F''(u)vw)(x) = f''(u(x))v(x)w(x)$$
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\end{aligned}
$$

No good. So $F''(u)$ is actually not in $\mathcal{L}^{(2)}(H)$ but in $\mathcal{L}^{(2)}(L_4,H)$ and hence $F \notin \mathcal{C}_b^2(H)$!

Nemytskij operator

But, by Sobolev's inequality, $\|\phi\|_{L_\infty} \leq C\|\phi\|_{H^s}$ if $s > d/2$, so that

$$\|h\|_{H^{-s}} = \sup_{\phi} \frac{\langle h, \phi \rangle}{\|\phi\|_{H^s}} \leq \sup_{\phi} \frac{\|h\|_{L_1}\|\phi\|_{L_\infty}}{\|\phi\|_{H^s}} \leq C\|h\|_{L_1}, \quad s > d/2$$

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Then

$$\|F''(u)vw\|_{H^{-s}} = \|f''(u)vw\|_{L_1} \le \|f''(u)\|_{L_\infty}\|v\|_{L_2}\|w\|_{L_2} \le |f|_{\mathcal{C}_b^2}\|v\|_H\|w\|_H$$
$$|F|_{\mathcal{C}_b^2(H,H^{-s})} := \sup_{u\in H}\|F''(u)\|_{\mathcal{L}^{(2)}(H,H^{-s})} \le |f|_{\mathcal{C}_b^2}$$

So $F''(u) \in \mathcal{L}^{(2)}(H, H^{-s})$ and $F \in \mathcal{C}_b^2(H, H^{-s})$.

Nemytskij operator

What about the diffusion coefficient $G(X)$?

$$g \in \mathcal{C}_{\mathrm{b}}^2(\mathbf{R}), \quad G\colon H \to \mathcal{L}_2^0(H), \quad (G(u)v)(x) = g(u(x))v(x)$$

$$\|G(u)\|^2_{\mathcal{L}_2^0(H)} = \|G(u)Q^{1/2}\|^2_{\mathcal{L}_2(H)} = \sum_{j=1}^{\infty} \|G(u)Q^{1/2}e_j\|_H^2 = \sum_{j=1}^{\infty} \gamma_j \|G(u)e_j\|_H^2$$

$$= \sum_{j=1}^{\infty} \gamma_j \|g(u)e_j\|^2_{L_2} \le \sum_{j=1}^{\infty} \gamma_j \|g(u)\|^2_{L_2} \|e_j\|^2_{L_\infty} \le \sum_{j=1}^{\infty} \gamma_j C \|1 + |u|\|^2_{L_2}$$

$$\le C \operatorname{Tr}(Q)(1 + \|u\|_{L_2})^2$$

Here $\|e_j\|_{L_2} = 1$ but we had to assume $\|e_j\|_{L_\infty} \le C$! Which is true sometimes, for example, $\sin(n\pi x)$.

Exercise. Compute $\|G'(u)\|_{\mathcal{L}(H,\mathcal{L}_2^0(H))}$. What about $\|G''(u)\|_{\mathcal{L}^{(2)}(H,\mathcal{L}_2^0(H))}$?

Implementation

Euler's method for the stochastic heat equation

$$\begin{cases} X^n \in S_h, \quad X^0 = P_h u_0 \\ X^n - X^{n-1} + \Delta t A_h X^n = P_h \Delta W^n \end{cases}$$

$$(X^n - X^{n-1}, \chi) + \Delta t(\nabla X^n, \nabla \chi) = (\underbrace{\Delta W^n}_{\in L_2(\Omega, \dot{H}^{-1})}, \chi), \quad \forall \chi \in S_h$$

$$X^n(x) = \sum_{k=1}^{N_h} X_k^n \phi_k(x), \quad \chi = \phi_j, \quad \{\phi_j\}_1^{N_h} \text{ finite element basis functions}$$

$$\sum_{k=1}^{N_h} X_k^n(\phi_k, \phi_j) + \Delta t \sum_{k=1}^{N_h} X_k^n(\nabla \phi_k, \nabla \phi_j) = \sum_{k=1}^{N_h} X_k^{n-1}(\phi_k, \phi_j) + (\Delta W^n, \phi_j)$$

$$\mathbf{M}\mathbf{X}^n + \Delta t \mathbf{K}\mathbf{X}^n = \mathbf{M}\mathbf{X}^{n-1} + \mathbf{b}^n$$

Implementation

How to simulate $\mathbf{b}_j^n = (\Delta W^n, \phi_j) = (W(t_n) - W(t_{n-1}), \phi_j)$?

Covariance of $\mathbf{b}^n$:

$$\mathbf{E}(\mathbf{b}_i^n \mathbf{b}_j^n) = \mathbf{E}\big((\Delta W^n, \phi_i)(\Delta W^n, \phi_j)\big) = \Delta t (Q\phi_i, \phi_j)$$

In other words:

$$\mathbf{E}(\mathbf{b}^n \otimes \mathbf{b}^n) = \Delta t \mathbf{Q}, \quad \mathbf{Q}_{ij} = (Q\phi_i, \phi_j).$$

This assumes that the action of the covariance operator is known (computable). For example, integral operator with known kernel: $(Qf)(x) = \int_{\mathcal{D}} q(x,y) f(y)\, \mathrm{d}y$.

Cholesky factorization: $\mathbf{Q} = \mathbf{L}\mathbf{L}^T$, expensive, but done only once.

Take $\mathbf{b}^n = \sqrt{\Delta t}\, \mathbf{L}\beta^n$, where $\beta^n \in \mathbf{R}^{N_h}$, $n = 1, 2, \ldots$, are $\mathcal{N}(0, \mathbf{I})$, that is, generate one random vector in each time step, the components are independent normally distributed random numbers.

Then

$$\begin{aligned}\mathbf{E}(\mathbf{b}^n \otimes \mathbf{b}^n) &= \mathbf{E}(\mathbf{b}^n (\mathbf{b}^n)^T) = \Delta t \mathbf{E}(\mathbf{L}\beta^n (\mathbf{L}\beta^n)^T) \\ &= \Delta t \mathbf{L}\mathbf{E}(\beta^n (\beta^n)^T)\mathbf{L}^T = \Delta t \mathbf{L}\mathbf{L}^T = \Delta t \mathbf{Q}\end{aligned}$$

Implementation

One situation where the action of Q is known is $Q = I$. Then $\mathbf{Q}_{ij} = (Q\phi_i, \phi_j) = (\phi_i, \phi_j)$, that is, $\mathbf{Q} = \mathbf{M}$, the mass matrix. It is sparse so the Cholesky factorization is not too expensive. It can also be approximated by the lumped mass matrix $\mathbf{M}_L$, which is diagonal and $\mathbf{M}_L^{1/2}$ is easily computed. Then $\mathbf{b}^n = \sqrt{\Delta t}\,\mathbf{M}_L^{1/2}\beta^n$ can be used.

But $Q = I$ is of no interest unless $d = 1$, as we have seen.

However, it can be used (also for $d \geq 1$) to generate noise increments ΔW with prescribed covariance from the Matérn class of covariance kernels. Let ΔW_I be a noise increment with $Q = I$ and solve the equation

$$(\kappa I - \Delta)^{(\nu+1)/2}\Delta W = \Delta W_I \quad \text{in } \mathcal{D}.$$

Then ΔW will have a covariance from the Matérn class with parameters κ, ν. Its finite element approximation will serve as the vector $\mathbf{b}$ above. But this equation is, in general, of fractional order $\nu + 1$ and it is therefore not straightforward to solve.

F. Lindgren and H. Rue, *An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach*, J. R. Statist. Soc. B (2011) **73**, Part 4, pp. 423-498.

Implementation

Another approach: truncate the orthogonal expansion (Karhunen–Loève expansion)

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{1/2} \beta_k(t) e_k \approx \sum_{k=1}^{M} \gamma_k^{1/2} \beta_k(t) e_k, \quad Q e_k = \gamma_k e_k.$$

The truncated expansion can be inserted in the finite element equation. This assumes that the eigenvectors of Q are known. The eigenvalues can be chosen with the desired rate of convergence $\gamma_k \to 0$.

Stochastic Cahn–Hilliard equation

The Cahn–Hilliard–Cook equation:

$$\begin{cases} \mathrm{d}u - \Delta v \, \mathrm{d}t = \mathrm{d}W & \text{in } \mathcal{D} \times (0, T]; \\ v + \Delta u - f(u) = 0 & \text{in } \mathcal{D} \times (0, T]; \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times (0, T]; \\ u(0) = u_0 & \text{in } \mathcal{D}. \end{cases} \tag{1}$$

Here $\mathcal{D} \subset \mathbf{R}^d$, $d \leq 3$, is a convex polygonal domain and

$$f(s) = F'(s), \quad F \text{ is a polynomial of degree 4}$$

$$F(s) \geq c_0 s^4 - c_1, \; c_0 > 0; \quad F''(s) \geq -\beta^2,$$

Typically: $F(s) = \frac{1}{4}(s^2 - \beta^2)^2$, $f(s) = s^3 - \beta^2 s$.
Further assumptions:

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- In order to preserve mass: $|\mathcal{D}|^{-1} \int_{\mathcal{D}} W(t)(x) \, \mathrm{d}x = 0$.
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Note: As $\tilde{F}(s) = F(s + s_0)$ has the same structural properties as F a change of variables $u \to u - |\mathcal{D}|^{-1} \int_{\mathcal{D}} u_0 \, \mathrm{d}x$ shows that we may assume that $|\mathcal{D}|^{-1} \int_{\mathcal{D}} u_0 \, \mathrm{d}x = 0$.

Abstract framework

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- W(t) – a Q-Wiener process in H with respect to $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$:

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$$\begin{cases} \mathrm{d}X + \big(A^2 X + A f(X)\big)\,\mathrm{d}t = \mathrm{d}W, & t > 0; \\ X(0) = X_0. \end{cases}$$

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- A is positive semidefinite on H and positive definite on $\dot{H}$ and hence A^{-1} is well-defined on $\dot{H}$.
- Fractional powers $A^{\alpha} v = \sum_{j=1}^{\infty} \lambda_j^{\alpha} \langle v, \varphi_j \rangle \varphi_j$ with corresponding spaces $\dot{H}^{\alpha} = D(A^{\frac{\alpha}{2}})$ and norms $|v|_{\alpha} = \|A^{\frac{\alpha}{2}} v\| = (\sum_{j=1}^{\infty} \lambda_j^{2\alpha} \langle v, \varphi_j \rangle^2)^{1/2}$.

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E(t)v = \mathrm{e}^{-tA^2}v &= \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2}\langle v, \varphi_j\rangle \varphi_j \\
&= \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2}\langle v, \varphi_j\rangle \varphi_j + \langle v, \varphi_0\rangle \varphi_0 \\
&= \mathrm{e}^{-tA^2}Pv + (I - P)v.
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$$\begin{cases} \dot{u} + A^2 u = 0, & t > 0; \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = \mathrm{e}^{-tA^2} v.$$

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$$\begin{aligned} E(t)v = \mathrm{e}^{-tA^2}v &= \sum_{j=0}^{\infty} \mathrm{e}^{-t\lambda_j^2}\langle v, \varphi_j\rangle \varphi_j \\ &= \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j^2}\langle v, \varphi_j\rangle \varphi_j + \langle v, \varphi_0\rangle \varphi_0 \\ &= \mathrm{e}^{-tA^2}Pv + (I-P)v. \end{aligned}$$

$$\begin{cases} \dot{u} + A^2 u = 0, \quad t > 0; \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = \mathrm{e}^{-tA^2}v.$$

- Analytic semigroup:

$$\|A^{\alpha}\mathrm{e}^{-tA^2}v\| \le Ct^{-\frac{\alpha}{2}}\|v\|, \quad v \in H, \ \alpha \ge 0.$$

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- Moreover:

$$\Big(\int_0^t s^{2\alpha}\|A^{2\alpha+1}\mathrm{e}^{-sA^2}v\|^2\,\mathrm{d}s\Big)^{1/2} \leq C\|v\|, \quad v \in H,\ \alpha \geq 0.$$

- $$\begin{cases} \dot{u} + A^2 u = f, \quad t > 0; \\ u(0) = v \end{cases} \qquad \Rightarrow u(t) = \mathrm{e}^{-tA^2}v + \int_0^t \mathrm{e}^{-(t-s)A^2}f(s)\,\mathrm{d}s.$$

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Weak solution:

$$\langle X(t), v\rangle - \langle X_0, v\rangle + \int_0^t \langle X(s), A^2 v\rangle\,\mathrm{d}s + \int_0^t \langle f(X(s)), Av\rangle\,\mathrm{d}s = \langle W(t), v\rangle$$

almost surely for all $v \in D(A^2)$, $t \in [0, T]$.

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A *mild solution* satisfies the equation:

$$X(t) = \mathrm{e}^{-tA^2} X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2} Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s).$$

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A *mild solution* satisfies the equation:

$$X(t) = e^{-tA^2}X_0 - \int_0^t e^{-(t-s)A^2} Af(X(s))\,ds + \int_0^t e^{-(t-s)A^2}\,dW(s).$$

Stochastic convolution:

$$W_A(t) = \int_0^t e^{-(t-s)A^2}\,dW(s).$$

$$X = Y + W_A, \quad Y(t) = e^{-tA^2}X_0 - \int_0^t e^{-(t-s)A^2} Af(Y(s) + W_A(s))\,ds$$

Existence, uniqueness and regularity

Theorem

If $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty$ *and* $|X_0|_1 < \infty$*, then there is a unique weak solution* X *of* (1)*. Furthermore, there is* $C_T > 0$ *such that*

$$\mathbf{E} \sup_{t\in[0,T]} |X(t)|_1^2 + \mathbf{E} \sup_{t\in[0,T]} \|X(t)\|_{L_4}^4 \le C_T.$$

In addition, X *is also a mild solution. Furthermore, for all* $\gamma \in [0, \frac{1}{2})$*, there is a finite nonnegative random variable* K *such that, almost surely,*

$$\sup_{t\neq s\in[0,T]} \frac{\|X(t)-X(s)\|}{|t-s|^\gamma} \le K.$$

Da Prato and Debussche, 1996

The finite element method

Spatial discretization:

- family of triangulations of $\mathcal{D}$: $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h

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- Conservation of mass:

$$W(t) \in \dot{H},\ X_0 \in \dot{H} \Rightarrow P_h X_0 \in \dot{H},\ X(t), X_h(t) \in \dot{H}.$$

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- note:

$$|v_h|_{1,h} = \|A_h^{1/2} v_h\| = \|\nabla v_h\| = \|A^{1/2} v_h\| = |v_h|_1, \quad v_h \in S_h \subset H^1(\mathcal{D}).$$

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$$\begin{cases} \mathrm{d}X_h + A_h^2 X_h \, \mathrm{d}t + A_h P_h f(X_h) \, \mathrm{d}t = P_h \, \mathrm{d}W, \quad t > 0, \\ X(0) = P_h X_0. \end{cases}$$

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- Mild formulation:

$$X_h(t) = \mathrm{e}^{-tA_h^2} P_h X_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2} A_h P_h f(X_h(s))\,\mathrm{d}s + \ int_0^t \mathrm{e}^{-(t-s)A_h^2} P_h \,\mathrm{d}W(s).$$

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- The fully discrete scheme:

$$\begin{cases} X_h^j - X_h^{j-1} + kA_h^2 X_h^j + kA_h P_h f(X_h^j) = P_h \Delta W^j, & t_j = jk,\ j = 1, 2, \ldots, N, \\ X_h^0 = P_h X_0. \end{cases} \tag{2}$$

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- Let $R_{k,h}^n = (I + k\,A_h^2)^{-n}$. Mild formulation:

$$X_h^n = R_{k,h}^n P_h X_0 - \sum_{j=1}^n R_{k,h}^{n-j+1} A_h P_h f(X_h^j) + \sum_{j=1}^n R_{k,h}^{n-j+1} P_h \Delta W^j.$$

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- Mass is conserved.

Linear CHC: regularity

$$\begin{cases} \mathrm{d}X + A^2 X \,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = 0 \end{cases}$$

Stochastic convolution: $X(t) = W_A(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$

Seminorms: $|v|_\beta = \|A^{\beta/2}v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^{\beta/2}\langle v, \varphi_j\rangle^2\Big)^{\frac{1}{2}}, \quad \dot{H}^\beta = D(A^{\beta/2}), \quad \beta \in \mathbf{R}$

Mean square: $\|v\|^2_{L_2(\Omega,\dot{H}^\beta)} = \mathbf{E}(|v|^2_\beta), \quad \beta \in \mathbf{R}$

Theorem

If $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty$ *for some* $\beta \geq 0$*, then*

$\|W_A(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS}, \quad t \geq 0.$

Regularity: proof

$$
\begin{aligned}
\|W_A(t)\|^2_{L_2(\Omega,\dot H^\beta)} &= \mathbf{E}\Big(\Big\|\int_0^t A^{\beta/2}E(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \|A^{\beta/2}E(s)Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \int_0^t \|AE(s)A^{(\beta-2)/2}Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \sum_{k=1}^{\infty}\int_0^t \|AE(s)A^{(\beta-2)/2}Q^{1/2}\phi_k\|^2\,\mathrm{d}s \\
&\le C\sum_{k=1}^{\infty}\|A^{(\beta-2)/2}Q^{1/2}\phi_k\|^2 \\
&= C\|A^{(\beta-2)/2}Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}
$$

$$\boxed{\int_0^t \|AE(s)v\|^2\,\mathrm{d}s \le C\|v\|^2}$$

Linear CHC: strong convergence

$$H = L_2(\mathcal{D}), \quad \dot{H} = \left\{ v \in H : \int_{\mathcal{D}} v \, \mathrm{d}x = 0 \right\}$$

Orthogonal projector $\quad P \colon H \to \dot{H}, \quad W(t) \in \dot{H}$

$$W_A(t) = \int_0^t E(t-s) \, \mathrm{d}W(s)$$

$$W_{A_h}(t) = \int_0^t E_h(t-s) P_h \, \mathrm{d}W(s)$$

$$\begin{aligned} W_{A_h}(t) - W_A(t) &= \int_0^t \big(E_h(t-s)P_h - E(t-s) \big) \, \mathrm{d}W(s) \\ &= \int_0^t F_h(t-s) \, \mathrm{d}W(s) \end{aligned}$$

Linear CHC: approximation of the semigroup

$$\begin{cases} \dot{u} + A^2 u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} \dot{u}_h + A_h^2 u_h = 0, & t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad\qquad u_h(t) = E_h(t)P_h v$$

Error: $F_h(t)v = E_h(t)P_h v - E(t)v$, norm: $|v|_\beta = \|A^{\beta/2}v\|$

Theorem

- $\|F_h(t)v\| \leq Ch^\beta |v|_\beta, \quad t \geq 0, \quad \beta \in [0,2]$
- $\Big(\int_0^t \|F_h(s)v\|^2 \, ds\Big)^{1/2} \leq Ch^\beta |\log(h)| \, |v|_{\beta-2}, \quad t \geq 0, \quad \beta \in [1,2]$

Note: the FEM is based on $(A_h)^2$ instead of $(A^2)_h$.

Linear CHC: strong convergence

Theorem

If $\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty$ *for some* $\beta \in [1,2]$*, then*

$$\|W_{A_h}(t) - W_A(t)\|_{L_2(\Omega,H)} \le Ch^{\beta}|\log(h)|\,\|A^{\frac{\beta-2}{2}}Q^{\frac{1}{2}}\|_{HS}, \quad t \ge 0.$$

Proof:

$$\begin{aligned}
&\|W_{A_h}(t) - W_A(t)\|^2_{L_2(\Omega,H)} \\
&\quad = \mathbf{E}\Big\|\int_0^t F_h(t-s)\,\mathrm{d}W(s)\Big\|^2 = \int_0^t \|F_h(t-s)Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&\quad = \sum_{j=1}^{\infty}\int_0^t \|F_h(t-s)Q^{1/2}\phi_j\|^2\,\mathrm{d}s \le C\sum_{j=1}^{\infty} h^{2\beta}|\log(h)|^2|Q^{1/2}\phi_j|^2_{\beta-2} \\
&\quad = Ch^{2\beta}|\log(h)|^2\sum_{j=1}^{\infty}\|A^{(\beta-2)/2}Q^{1/2}\phi_j\|^2 \\
&\quad = Ch^{2\beta}|\log(h)|^2\|A^{(\beta-2)/2}Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}$$

Linear CHC: strong convergence

The assumption is: $\|A^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty, \quad \beta \in [1,2]$.

- $\mathrm{Tr}(Q) = \|Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$: $\quad \beta = 2$.
- $Q = I$, "white noise":

$$\|A^{\frac{\beta-2}{2}}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} \lambda_j^{\beta-2} \sim \sum_{j=1}^{\infty} j^{(\beta-2)2/d} < \infty$$

if $\beta < 2 - d/2$. Thus: $d = 1$, $\beta \in [1, 3/2)$.

Linear CHC: strong convergence

Larsson and Mesforush, IMAJNA (2011).
Euler timestepping is also studied here.

Kossioris and Zouraris, M2AN (2010) (1-D)

Fully discrete, nonlinear: main result

Theorem

Suppose that $\|A^{1/2}Q^{1/2}\|_{HS} < \infty$ *(*$\beta = 3$*) and that there is* $L > 0$ *such that*

$$|X_0|_1 + |X_h^0|_1 + \mathcal{F}(X_h^0) + |A_h X_h^0 + P_h f(X_h^0)|_1 \leq L, \quad 0 < h < h_0,$$

where $\mathcal{F}(u) = \int_{\mathcal{D}} F(u(x))\,dx$*. Then*

$$\lim_{h,k\to 0} \mathbf{E} \sup_{t_n \in [0,T]} \|X(t_n) - X_h^n\|^2 = 0.$$

Thus, we show strong convergence, but there is no error estimate, so the rate of convergence is not obtained.

Local Lipschitz condition

The semilinear term $Af(u)$ is not globally Lipschitz and does not have a linear growth. Instead we have local Lipschitz conditions, for example,

$$
\begin{aligned}
|Af(u) - Af(v)|_{-3} &= \|f(u) - f(v)\|_{-1} \le C\|f(u) - f(v)\|_{L_{6/5}} \\
&\le C\|(1 + u^2 + v^2)(u - v)\|_{L_{6/5}} \le C(1 + \|u\|_{L_6}^2 + \|v\|_{L_6}^2)\|u - v\|_{L_2} \\
&\le C(1 + |u|_1^2 + |v|_1^2)\|u - v\|_{L_2}.
\end{aligned}
$$

This can be used in the mild formulation, for example,

$$
\begin{aligned}
&\mathbf{E}\Big\| \int_0^t \mathrm{e}^{-(t-s)A^2} A(f(X(s)) - f(X_h(s)))\,\mathrm{d}s \Big\| \\
&= \mathbf{E}\Big\| \int_0^t A^{3/2}\mathrm{e}^{-(t-s)A^2} A^{-1/2}(f(X(s)) - f(X_h(s)))\,\mathrm{d}s \Big\| \\
&\le \int_0^t \mathbf{E}\|A^{3/2}\mathrm{e}^{-(t-s)A^2} A^{-1/2}(f(X(s)) - f(X_h(s)))\|\,\mathrm{d}s \\
&\le \int_0^t \|A^{3/2}\mathrm{e}^{-(t-s)A^2}\|_{\mathcal{L}(H)} \mathbf{E}\|f(X(s)) - f(X_h(s))\|_{-1}\,\mathrm{d}s \\
&\le C\int_0^t (t-s)^{-3/4}\mathbf{E}[(1 + |X(s)|_1^2 + |X_h(s)|_1^2)\|X(s) - X_h(s)\|]\,\mathrm{d}s
\end{aligned}
$$

Local Lipschitz condition

$$\begin{aligned}\mathbf{E}\Big\|\int_0^t \mathrm{e}^{-(t-s)A^2}A(f(X(s))-f(X_h(s)))\,\mathrm{d}s\Big\| \\ \le C\int_0^t (t-s)^{-3/4}\mathbf{E}[(1+|X(s)|_1^2+|X_h(s)|_1^2)\|X(s)-X_h(s)\|]\,\mathrm{d}s\end{aligned}$$

Here, by Hölder's inequality,

$$\begin{aligned}\mathbf{E}[(1+|X(s)|_1^2+|X_h(s)|_1^2)\|X(s)-X_h(s)\|] \\ \le (1+\mathbf{E}|X(s)|_1^4+\mathbf{E}|X_h(s)|_1^4)^{1/2}(\mathbf{E}\|X(s)-X_h(s)\|^2)^{1/2}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{E}\Big\|\int_0^t \mathrm{e}^{-(t-s)A^2}A(f(X(s))-f(X_h(s)))\,\mathrm{d}s\Big\| \\ \le C_t \sup_{s\in[0,t]}(1+\mathbf{E}|X(s)|_1^4+\mathbf{E}|X_h(s)|_1^4)^{1/2}\sup_{s\in[0,t]}(\mathbf{E}\|X(s)-X_h(s)\|^2)^{1/2}.\end{aligned}$$

This indicates the need for moment bounds for $X(t)$, $X_h(t)$ and X_h^n.

Local Lipschitz condition

$$\mathbf{E}\Big\|\int_0^t \mathrm{e}^{-(t-s)A^2}A(f(X(s))-f(X_h(s)))\,\mathrm{d}s\Big\|$$
$$\leq C\int_0^t (t-s)^{-3/4}\mathbf{E}[(1+|X(s)|_1^2+|X_h(s)|_1^2)\|X(s)-X_h(s)\|]\,\mathrm{d}s$$

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$$\leq (1+\mathbf{E}|X(s)|_1^4+\mathbf{E}|X_h(s)|_1^4)^{1/2}(\mathbf{E}\|X(s)-X_h(s)\|^2)^{1/2}.$$

Thus,

$$\mathbf{E}\Big\|\int_0^t \mathrm{e}^{-(t-s)A^2}A(f(X(s))-f(X_h(s)))\,\mathrm{d}s\Big\|$$
$$\leq C_t \sup_{s\in[0,t]}(1+\mathbf{E}|X(s)|_1^4+\mathbf{E}|X_h(s)|_1^4)^{1/2}\sup_{s\in[0,t]}(\mathbf{E}\|X(s)-X_h(s)\|^2)^{1/2}.$$

This indicates the need for moment bounds for $X(t)$, $X_h(t)$ and X_h^n.
However, the norm on the left is $L_1(\Omega,H)$ while on the right we have $\|X(s)-X_h(s)\|_{L_2(\Omega,H)}$. We need to have the same $L_2(\Omega,H)$ on both sides. But then we must have $\|X\|_{L_\infty(\Omega\times[0,t],\dot{H}^1)}$ and $\|X_h\|_{L_\infty(\Omega\times[0,t],\dot{H}^1)}$ in the Lipschitz constant.
This is not possible, but we will find a work-around, which yields convergence but no error estimate.

Main obstacles

The proof of moment bounds must take advantage of structural properties of $f(u)$, and must therefore be based on the weak formulation instead of the mild formulation. Some difficulties:

- It follows from the conditions on F that

$$\langle f(x), x\rangle \geq -C_0 - C_1\|x\|^2,$$

  and even

$$\langle A^{\frac{1}{2}} f(x), A^{\frac{1}{2}} x\rangle = \langle \nabla f(x), \nabla x\rangle \geq -c|x|_1^2.$$

Main obstacles

- Matters are worse for the finite element approximation. While

$$\langle P_h f(v_h), v_h \rangle = \langle f(v_h), v_h \rangle \geq -C_0 - C_1 \|v_h\|^2, \quad v_h \in S_h, \tag{3}$$

  unfortunately

$$\langle A_h^{\frac{1}{2}} P_h f(v_h), A_h^{\frac{1}{2}} v_h \rangle = \langle A^{\frac{1}{2}} P_h f(v_h), A^{\frac{1}{2}} v_h \rangle \not\geq -c|v_h|_1^2, \quad v_h \in S_h, \tag{4}$$

The operators P_h, e^{-tA^2}, $\mathrm{e}^{-tA_h^2}$ act globally in $\mathcal{D}$ and do not preserve the pointwise structural properties of $f(u)$. Therefore, they cannot be exploited in connection with the mild formulation, and the P_h must be removed as in (3). This cannot be done in (4).

Deterministic error estimates

- The following error estimates are readily available in the literature:

$$\|(E(t_n) - R_{k,h}^n)P_h v\| \le C(h^\beta + k^{\beta/4})|v|_\beta, \quad t_n \ge 0,\ \beta \in [0,2]$$

$$\|(E(t_n) - R_{k,h}^n)P_h v\| \le C(h^\beta + k^{\beta/4})t_n^{-(\beta-\gamma)/4}|v|_\gamma, \quad t > 0,$$
$$\gamma \in [-1,1],\ \max(0,\gamma) \le \beta \le 2.$$

Deterministic error estimates

- The following error estimates are readily available in the literature:

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$$\|(E(t_n) - R_{k,h}^n)P_h v\| \le C(h^\beta + k^{\beta/4})t_n^{-(\beta-\gamma)/4}|v|_\gamma, \quad t > 0,$$
$$\gamma \in [-1,1],\ \max(0,\gamma) \le \beta \le 2.$$

- These lead to an error bound for the stochastic convolution:

Lemma

Let $\gamma \in (0, \frac{1}{2}]$, $\beta \in [1,2]$, and $p > \frac{2}{\gamma}$. Then there is $C = C(p,\gamma,T)$ such that

$$\left(\mathbf{E}\Big(\sup_{t_n \in [0,T]} \|W_A(t_n) - W_{A_h}^n\|^p\Big)\right)^{1/p} \le C(h^\beta + k^{\beta/4})\|A^{(\beta-2)/2+\gamma}Q^{1/2}\|_{HS}.$$

The proof is based on a "factorization argument" from Da Prato–Zabczyk.
We will use $\beta = 2$, $\gamma = \frac{1}{2}$, $\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}$.

Deterministic Cahn–Hilliard equation

Gradient flow in $\dot{H}^{-1}$ for the energy functional:

$$\begin{aligned} J(u) &= \tfrac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x \\ &= \tfrac{1}{2}|u|_1^2 + \mathcal{F}(u), \quad u \in \dot{H}^1, \quad F(s) = \tfrac{1}{4}s^4 - \tfrac{1}{2}s^2 \\ J'(u) &= Au + f(u) \end{aligned}$$

Deterministic case: $\dot{u} + A(Au + f(u)) = 0$, that is

$$\dot{u} + AJ'(u) = 0, \quad \text{or } \dot{u} = -AJ'(u).$$

Multiply by $A^{-1}\dot{u}$:

$$\begin{aligned} &\langle \dot{u}, A^{-1}\dot{u}\rangle + \langle AJ'(u), A^{-1}\dot{u}\rangle = 0 \\ &|\dot{u}|_{-1}^2 + D_t J'(u) = 0 \\ &D_t J(u) = -|\dot{u}|_{-1}^2 \le 0 \\ &J(u(t)) \le J(u_0), \quad t \ge 0 \quad \text{(Lyapunov functional)} \end{aligned}$$

Note: $J(u(t))$ is equivalent to $|u(t)|_1^2 + \|u(t)\|_{L_4(\mathcal{D})}^4$, so we get a bound for these norms.

Deterministic Cahn–Hilliard equation

Another proof.

$$\dot{u} + A^2 u + Af(u) = 0,$$

Do not eliminate the chemical potential $v = Au + f(u)$:

$$\begin{cases} \dot{u} + Av = 0, \\ v = Au + f(u) \end{cases}$$

Multiply by v:

$$\langle \dot{u}, v \rangle + \langle Av, v \rangle = 0$$

But we have

$$J(u) = \frac{1}{2}\|A^{1/2}u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x$$

$$J'(u) = Au + f(u) = v$$

$$D_t J(u) = \langle J'(u), \dot{u} \rangle = \langle v, \dot{u} \rangle$$

Therefore:

$$D_t J(u) + |v|_1^2 = 0$$

$$J(u(t)) + \int_0^t |v|_1^2\,\mathrm{d}s = J(u_0), \quad t \geq 0$$

$$J(u(t)) \leq J(u_0), \quad t \geq 0 \quad \text{(Lyapunov functional)}$$

Time continuous stochastic case:

Theorem

If $\|A^{1/2}Q^{1/2}\|_{HS}^2 < \infty$, *then*

$$\mathbf{E}\big[J(X(t))\big] \le C(t), \quad \mathbf{E}\big[J(X_h(t))\big] \le C(t), \quad t \ge 0.$$

Moreover,

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\Big(|X(t)|_1^2 + \|X(t)\|_{L_4}^4\Big)\Big] \le K_T,$$

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\Big(|X_h(t)|_1^2 + \|X_h(t)\|_{L_4}^4\Big)\Big] \le K_T.$$

Cahn–Hilliard–Cook equation

Proof for $J(X_h(t))$:

$$
\begin{aligned}
J(u_h) &= \frac{1}{2}\|A_h^{1/2}u_h\|^2 + \int_{\mathcal{D}} F(u_h)\,\mathrm{d}x \\
J'(u_h) &= A_h u_h + P_h f(u_h) \\
J''(u_h) &= A_h + P_h[f'(u_h)\cdot]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}X_h &= -A_h(A_h X_h + P_h f(X_h))\,\mathrm{d}t + P_h\,\mathrm{d}W \\
&= -A_h J'(X_h)\,\mathrm{d}t + P_h\,\mathrm{d}W
\end{aligned}
$$

Itô's formula: (with $Q_h = P_h Q P_h$)

$$
\begin{aligned}
J(X_h(t)) &= J(X_h(0)) + \int_0^t \langle J'(X_h(s)), \mathrm{d}X_h(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s \\
&= J(P_h X_0) - \int_0^t \langle J'(X_h(s)), A_h J'(X_h(s))\rangle\,\mathrm{d}s \\
&\quad + \int_0^t \langle J'(X_h(s)), P_h\,\mathrm{d}W(s)\rangle + \frac{1}{2}\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s.
\end{aligned}
$$

Cahn-Hilliard-Cook equation

$$
\begin{aligned}
&\mathbf{E}\big[J(X_h(t))\big] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\big[J(P_h X_0)\big] + \underbrace{\mathbf{E}\Big[\int_0^t \langle J'(X_h(s)), P_h \,\mathrm{d}W(s)\rangle\Big]}_{=0} \\
&\qquad + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\big[J(P_h X_0)\big] + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}(A_h Q_h)\,\mathrm{d}s\Big] + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}([f'(X_h(s))\cdot]Q_h)\,\mathrm{d}s\Big]
\end{aligned}
$$

Proof completed by bounding these terms and using Gronwall's lemma.
For example, $\mathrm{Tr}(A_h Q_h) \le C\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}$.
For the second variant: take $\sup_{t\in[0,T]}$ before $\mathbf{E}$.

Cahn-Hilliard-Cook equation

$$
\begin{aligned}
&\mathbf{E}\big[J(X_h(t))\big] + \mathbf{E}\Big[\int_0^t |J'(X_h(s))|_1^2 \,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\big[J(P_h X_0)\big] + \underbrace{\mathbf{E}\Big[\int_0^t \langle J'(X_h(s)), P_h \,\mathrm{d}W(s)\rangle\Big]}_{=0} \\
&\qquad + \frac{1}{2}\mathbf{E}\Big[\int_0^t \mathrm{Tr}(J''(X_h(s))Q_h)\,\mathrm{d}s\Big] \\
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\end{aligned}
$$

Proof completed by bounding these terms and using Gronwall's lemma.
For example, $\mathrm{Tr}(A_h Q_h) \le C\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}$.
For the second variant: take $\sup_{t\in[0,T]}$ before $\mathbf{E}$.

This was used to prove convergence (without rate) for the spatially semidiscrete approximation. Kovács–Larsson–Mesforush, 2011.

Now we want to imitate this for the fully discrete approximation.
Difficulty: no Itô formula.

Moment bound

Fully discrete scheme:

$$X_h^j - X_h^{j-1} + kA_h(A_hX_h^j + P_hf(X_h^j)) = P_h\Delta W^j$$

Define $Y_h^j = A_hX_h^j + P_hf(X_h^j)$ (discrete chemical potential).

$$X_h^j - X_h^{j-1} + kA_hY_h^j = P_h\Delta W^j$$

Theorem

Let $p \geq 1$. If $\|A^{1/2}Q^{1/2}\|_{HS} \leq L$ and

$$|X_h^0|_1 + \mathcal{F}(X_h^0) + |Y_h^0|_1 \leq L,$$

then there exists $C, k_0 > 0$, depending on p, T and L, such that

$$\mathbf{E}\sup_{1\leq j\leq N}|X_h^j|_1^{2p} + \mathbf{E}\sup_{1\leq n\leq N}\mathcal{F}(X_h^n)^p + \mathbf{E}\Big(\sum_{j=1}^N k|Y_h^j)|_1^2\Big)^p \leq C, \quad k \leq k_0.$$

Moment bound

Fully discrete scheme:

$$X_h^j - X_h^{j-1} + kA_h(A_h X_h^j + P_h f(X_h^j)) = P_h \Delta W^j$$

Define $Y_h^j = A_h X_h^j + P_h f(X_h^j)$ (discrete chemical potential).

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Theorem

Let $p \geq 1$. If $\|A^{1/2}Q^{1/2}\|_{HS} \leq L$ and

$$|X_h^0|_1 + \mathcal{F}(X_h^0) + |Y_h^0|_1 \leq L,$$

then there exists $C, k_0 > 0$, depending on p, T and L, such that

$$\mathbf{E} \sup_{1\leq j\leq N} |X_h^j|_1^{2p} + \mathbf{E} \sup_{1\leq n\leq N} \mathcal{F}(X_h^n)^p + \mathbf{E}\Big(\sum_{j=1}^{N} k|Y_h^j)|_1^2\Big)^p \leq C, \quad k \leq k_0.$$

Proof. Multiply by $A_h^{-1}(X_h^j - X_h^{j-1})$ does not work:

$$\langle X_h^j - X_h^{j-1}, A_h^{-1}(X_h^j - X_h^{j-1})\rangle + k\langle A_h(A_h X_h^j + P_h f(X_h^j)), A_h^{-1}(X_h^j - X_h^{j-1})\rangle$$
$$= \langle P_h \Delta W^j, A_h^{-1}(X_h^j - X_h^{j-1})\rangle$$

$$|X_h^j - X_h^{j-1}|_{-1,h}^2 + k\big(J(X_h^j) - J(X_h^{j-1})\big) + \ldots = \tfrac{1}{2}\underbrace{|P_h \Delta W^j|_{-1,h}^2}_{\approx k} + \tfrac{1}{2}|X_h^j - X_h^{j-1}|_{-1,h}^2$$

Proof, continued

$$X_h^j - X_h^{j-1} + kA_hY_h^j = P_h\Delta W^j$$

Multiply by Y_h^j:

$$\langle X_h^j - X_h^{j-1}, Y_h^j\rangle + k|Y_h^j|_1^2 = \langle Y_h^j, P_h\Delta W^j\rangle$$

Here:

$$\begin{aligned}\langle X_h^j - X_h^{j-1}, Y_h^j\rangle &= \langle X_h^j - X_h^{j-1}, A_hX_h^j\rangle + \langle X_h^j - X_h^{j-1}, P_hf(X_h^j)\rangle \\ &\geq \tfrac{1}{2}\big(|X_h^j|_1^2 - |X_h^{j-1}|_1^2 + |X_h^j - X_h^{j-1}|_1^2\big) \\ &\quad + \mathcal{F}(X_h^j) - \mathcal{F}(X_h^{j-1}) - \beta^2\|X_h^j - X_h^{j-1}\|^2 \\ &= J(X_h^j) - J(X_h^{j-1}) + \tfrac{1}{2}|X_h^j - X_h^{j-1}|_1^2 - \beta^2\|X_h^j - X_h^{j-1}\|^2\end{aligned}$$

Sum up (with $\Delta X_h^j := X_h^j - X_h^{j-1}$):

$$J(X_h^n) + k\sum_{j=1}^n |Y_h^j|_1^2 + \frac{1}{2}\sum_{j=1}^n |\Delta X_h^j|_1^2 \leq J(X_h^0) + \sum_{j=1}^n \langle Y_h^j, P_h\Delta W^j\rangle + \beta^2\sum_{j=1}^n \|\Delta X_h^j\|^2$$

Here:

$$\sum_{j=1}^n \langle Y_h^j, P_h\Delta W^j\rangle = \sum_{j=1}^n \langle Y_h^{j-1}, P_h\Delta W^j\rangle + \sum_{j=1}^n \langle \Delta Y_h^j, P_h\Delta W^j\rangle$$

Raise to power p, take $\sup_{1\leq n\leq N}$ and then $\mathbf{E}$.

Proof, continued

Most difficult term:

$$\sum_{j=1}^{n} \langle \Delta Y_h^j, P_h \Delta W^j \rangle = \sum_{j=1}^{n} \langle A_h (X_h^j - X_h^{j-1}) + P_h (f(X_h^j) - f(X_h^{j-1})), P_h \Delta W^j \rangle$$

Delicate calculation of the Lipschitz constant of f using auxiliary moment bounds...

Auxiliary moment bound 1

Lemma

Let $p \geq 1$*. If* $\|Q^{1/2}\|_{HS} < \infty$ *and* $|X_0|_{-1,h} < L$ *then there exists a* $C > 0$ *and* $k_0 > 0$ *such that, for* $0 < k < k_0$*,*

$$\mathbf{E} \sup_{1 \leq j \leq N} |X_h^j|_{-1,h}^{2p} \leq C,$$

$$\mathbf{E} \left(\sum_{j=1}^{n} \left(|X_h^j - X_h^{j-1}|_{-1,h}^2 + k |X_h^j|_1^2 \right) \right)^p \leq C.$$

Auxiliary moment bound 1

Lemma

Let $p \geq 1$. If $\|Q^{1/2}\|_{HS} < \infty$ and $|X_0|_{-1,h} < L$ then there exists a $C > 0$ and $k_0 > 0$ such that, for $0 < k < k_0$,

$$\mathbf{E} \sup_{1 \leq j \leq N} |X_h^j|_{-1,h}^{2p} \leq C,$$

$$\mathbf{E} \left(\sum_{j=1}^{n} \left(|X_h^j - X_h^{j-1}|_{-1,h}^2 + k|X_h^j|_1^2 \right) \right)^p \leq C.$$

Proof. Take inner products with $A_h^{-1} X_h^j$ and use

$$\langle P_h f(v_h), v_h \rangle = \langle f(v_h), v_h \rangle \geq -C_0 - C_1 \|v_h\|^2.$$

Apply Gronwall's Lemma to finish.

Auxiliary moment bound 2

Lemma

Suppose that $\|Q^{1/2}\|_{HS} < \infty$ *and* $\|X^0\| + |X_h^0|_{-1,h} < L$. *Then, for every* $\epsilon, \delta > 0$ *and* $p \geq 1$, *there is* $C = C(T, \epsilon, \delta, p, X^0, \|Q^{1/2}\|_{HS}) > 0$ *and* $K = K(T, p) > 0$ *and* $k_0 = k_0(\epsilon, p)$ *such that for* $0 < k < k_0$,

$$\mathbf{E} \sup_{1 \leq j \leq N} \|X_h^j\|^{2p} + \mathbf{E} \left(\sum_{j=1}^{N} \|X_h^j - X_h^{j-1}\|^2 \right)^p \leq C + K\delta \mathbf{E} \left(\sum_{j=1}^{N} k|Y_h^j)|_1^2 \right)^{\frac{1+\epsilon}{2} p} .$$

Auxiliary moment bound 2

Lemma

Suppose that $\|Q^{1/2}\|_{HS} < \infty$ *and* $\|X^0\| + |X_h^0|_{-1,h} < L$*. Then, for every* $\epsilon, \delta > 0$ *and* $p \geq 1$*, there is* $C = C(T, \epsilon, \delta, p, X^0, \|Q^{1/2}\|_{HS}) > 0$ *and* $K = K(T, p) > 0$ *and* $k_0 = k_0(\epsilon, p)$ *such that for* $0 < k < k_0$*,*

$$\mathbf{E} \sup_{1 \leq j \leq N} \|X_h^j\|^{2p} + \mathbf{E}\left(\sum_{j=1}^{N} \|X_h^j - X_h^{j-1}\|^2\right)^p \leq C + K\delta \mathbf{E}\left(\sum_{j=1}^{N} k|Y_h^j)|_1^2\right)^{\frac{1+\epsilon}{2}p}.$$

Proof. Taking inner products with X_h^j but keep in mind that

$$\langle A_h^{\frac{1}{2}} P_h f(v_h), A_h^{\frac{1}{2}} v_h \rangle \ngeq -c|v_h|_1^2, \quad v_h \in S_h.$$

However we may use from the previous result that with any $q \geq 1$,

$$\mathbf{E}\left(\sum_{j=1}^{n} k|X_h^j|_1^2\right)^q \leq C.$$

Uniform bounds

We now have

$$\mathbf{E} \sup_{1\le j\le N} \left(|X_h^j|_1^2 + \|X_h^j\|_{L_4}^4\right) \le K_T$$

$$\mathbf{E} \sup_{t\in[0,T]} |X(t)|_1^2 + \mathbf{E} \sup_{t\in[0,T]} \|X(t)\|_{L_4}^4 \le K_T$$

$$\mathbf{E} \sup_{t_n\in[0,T]} \frac{\|W_A(t_n) - W_{A_h}^n\|^2}{(h^2 + k^{1/2})^2} \le K_T$$

Therefore, by Chebyshev's inequality, for any ϵ there is $\Omega_{h,k}^\epsilon \subset \Omega$ with $\mathbf{P}(\Omega_{h,k}^\epsilon) > 1 - \epsilon$, such that

$$\sup_{1\le j\le N} \left(|X_h^j|_1^2 + \|X_h^j\|_{L_4}^4\right) \le K_{\epsilon,T}$$

$$\sup_{t\in[0,T]} |X(t)|_1^2 + \mathbf{E} \sup_{t\in[0,T]} \|X(t)\|_{L_4}^4 \le K_{\epsilon,T}$$

$$\sup_{t_n\in[0,T]} \frac{\|W_A(t_n) - W_{A_h}^n\|^2}{(h^2 + k^{1/2})^2} \le K_{\epsilon,T}$$

on $\Omega_{h,k}^\epsilon$. We can now estimate the error using deterministic methods on $\Omega_{h,k}^\epsilon$.

Chebyshev's inequality

Let F be a random variable with $\mathbf{E}[F] \leq K$.

Chebyshev's inequality gives, for every $\alpha > 0$,

$$\mathbf{P}\Big(\big\{\omega \in \Omega : F > \alpha\big\}\Big) \leq \frac{1}{\alpha}\mathbf{E}\big[F\big] \leq \frac{K}{\alpha}.$$

We choose $\alpha = \epsilon^{-1}K$ and set $\Omega_\epsilon = \big\{\omega \in \Omega : F \leq \epsilon^{-1}K\big\}$. Then

$$\mathbf{P}(\Omega_\epsilon) = 1 - \mathbf{P}\Big(\big\{\omega \in \Omega : F > \alpha\big\}\Big) \geq 1 - \epsilon.$$

Recall that we have

$$\mathbf{E} \sup_{1 \leq j \leq N} \left(|X_h^j|_1^2 + \|X_h^j\|_{L_4}^4\right) \leq K_T$$

$$\mathbf{E} \sup_{t \in [0,T]} |X(t)|_1^2 + \mathbf{E} \sup_{t \in [0,T]} \|X(t)\|_{L_4}^4 \leq K_T$$

$$\mathbf{E} \sup_{t_n \in [0,T]} \frac{\|W_A(t_n) - W_{A_h}^n\|^2}{(h^2 + k^{1/2})^2} \leq K_T$$

We apply Chebyshev with $F = \sup_{1 \leq j \leq N}\left(|X_h^j|_1^2 + \|X_h^j\|_{L_4}^4\right)$, and so on.

Convergence

Theorem

Suppose that $\|A^{1/2}Q^{1/2}\|_{HS} < \infty$ *and that*

$$|X_h^0|_1 + \mathcal{F}(X_h^0) + |Y_h^0|_1 + |X_0|_2 \leq L.$$

Let $h, k > 0$ *small and* $0 < \epsilon, \delta < 1$*. Then, there is* $\Omega_{h,k}^{\epsilon} \subset \Omega$ *with* $\mathbf{P}(\Omega_{h,k}^{\epsilon}) > 1 - \epsilon$*, and* $C = C(T, L, \epsilon, \delta)$ *such that for all* $\omega \in \Omega_{h,k}^{\epsilon}$*,*

$$\|X(t_n) - X_h^n\| \leq C(h^{2(1-\delta)} + k^{1/2(1-\delta)}), \quad t_n \in [0, T].$$

Proof. The proof is based on the mild formulation, using the moment bounds, the error estimate for the stochastic convolution, and Gronwall's inequality on $\Omega_{h,k}^{\epsilon}$.
Remark. If only

$$|X_h^0|_1 + \mathcal{F}(X_h^0) + |Y_h^0|_1 + |X_0|_1 \leq L,$$

then

$$\|X(t_n) - X_h^n\| \leq C(h + k^{1/4}), \quad t_n \in [0, T].$$

Proof of the main result

The main result:

$$\lim_{h,k\to 0} \mathbf{E} \sup_{t_n\in[0,T]} \|X(t_n) - X_h^n\|^2 = 0.$$

Proof:

$$\begin{aligned}
\mathbf{E} \sup_{0\le t_n\le T} \|X(t_n) - X_h^n\|^2 &\le \int_{\Omega_{h,k}^{\epsilon}} \sup_{0\le t_n\le T} \|X(t_n) - X_h^n\|^2 \, d\mathbf{P} \\
&\quad + 2\int_{(\Omega_{h,k}^{\epsilon})^c} \sup_{0\le t_n\le T} \left(\|X(t_n)\|^2 + \|X_h^n\|^2\right) \, d\mathbf{P} \\
&\le C_\epsilon(h^2 + k^{1/2}) + 4\epsilon^{1/2} \left(\int_{(\Omega_{h,k}^{\epsilon})^c} \sup_{0\le t_n\le T} \left(\|X(t_n)\|^4 + \|X_h^n\|^4\right) \, d\mathbf{P} \right)^{1/2} \\
&\le C_\epsilon(h^2 + k^{1/2}) + 4\epsilon^{1/2} \left(\mathbf{E} \sup_{0\le t_n\le T} \left(\|X(t_n)\|^4 + \|X_h^n\|^4\right) \right)^{1/2} \\
&\le C_\epsilon(h^2 + k^{1/2}) + 4\epsilon^{1/2} |\mathcal{D}|^{1/2} \left(\mathbf{E} \sup_{0\le t_n\le T} \left(\|X(t_n)\|_{L_4}^4 + \|X_h^n\|_{L_4}^4\right) \right)^{1/2} \\
&\le C_\epsilon(h^2 + k^{1/2}) + 4\epsilon^{1/2} |\mathcal{D}|^{1/2} K_T.
\end{aligned}$$