Hand-in exercise
SPDE School at Chalmers, 7-10 March 2017

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The task consists in proving well-posedness of a non-linear version of the stochastic heat equation. For details on the notation used, see the lecture slides on the web-page of the course.

The SPDE is given by

$$dX + AX\,dt = F(X)dt + G(X)dW. \tag{1}$$

The case $F(X) \equiv 0$ and $G(X) \equiv I$ gives the standard stochastic heat equation. The non-linear operators F and G are Lipschitz continuous with respect to the relevant norms, i.e.,

$$\|F(u) - F(v)\|_H \leq C\|u - v\|_H, \quad \text{and} \quad \|G(u) - G(v)\|_{\mathcal{L}_2^0(H)} \leq C\|u - v\|_H. \tag{2}$$

Recall the operator norm

$$\|T\|^2_{\mathcal{L}_2^0(H)} = \sum_{j=1}^{\infty} \|TQ^{1/2}e_j\|^2_H. \tag{3}$$

We are looking for a mild solution

$$X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s))ds + \int_0^t E(t-s)G(X(s))dW(s). \tag{4}$$

To solve this equation we can use that it is of the form of a fixed point formulation

$$X = \mathcal{F}(X). \tag{5}$$

The task then consists in showing that

$$\|\mathcal{F}(X) - \mathcal{F}(Y)\|_{L^\infty([0,T],L^2(\Omega,\dot{H}^\beta))} \leq \kappa \|X - Y\|_{L^\infty([0,T],L^2(\Omega,\dot{H}^\beta))} \tag{6}$$

for some $\kappa < 1$. Notice that κ typically depends on T, so the condition $\kappa < 1$ may constrain T to be small.

Hint: study the linear case $F(X) \equiv 0$ and $G(X) \equiv I$ in the lecture slides.

GOOD LUCK!