On the Lifting of the Dirac Elements in the Higson-Kasparov Theorem

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“Analysis, Noncommutative Geometry, Operator Algebras”
Gothenburg University, Sweden, 12-16/6 2017
Plan of the Talk

Main Topic: The Higson-Kasparov Theorem (2001) = The Baum-Connes Conjecture for a-T-menable groups

- What is the main idea for proving this theorem
- What kind of interesting analysis comes into play (Non-commutative functional calculus: definitions and corrections)
- Simplifications in the lifting argument from $E$-theory to $KK$-theory
# a-T-menable Groups

## Definition: a-T-menable groups
A second countable locally compact group is **a-T-menable** if it acts metrically properly and affine isometrically on a (real) Hilbert space.

## Examples of a-T-menable groups
- All compact groups.
- **All amenable groups** (e.g. abelian groups).
- Groups which act properly on trees (e.g. free groups $F_n$).
- All closed subgroups of $SO(n, 1)$ or $SU(n, 1)$.

## Non-examples
- $Sp(n, 1)$ for $n \geq 2$, $SL(n, \mathbb{Z})$, $SL(n, \mathbb{R})$ for $n \geq 3$ (Property (T));
- $SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ (relative Property (T)).
BCC

- $G$: a second countable, locally compact topological group.
- $\mathcal{E}G$: a classifying space for proper actions of $G$.
- $A$: a separable $G$-$C^*$-algebra.
- $A \rtimes_r G$: the reduced crossed product of $A$.

The Baum-Connes Conjecture with Coefficients (BCC), 1991

$$RKK^G_*(\mathcal{E}G, A) \cong K_*(A \rtimes_r G) \quad (\ast = 0, 1)$$

- LHS = Equivariant K-homology of $\mathcal{E}G$ with coefficient in $A$.
- RHS = K-theory of $A \rtimes_r G$.

$G$ satisfies BCC $\Rightarrow$ All closed subgroups of $G$ satisfy BCC.

BCC $\Rightarrow$ The Baum-Connes Conjecture (when $A = \mathbb{C}$).
**BCC and Higson-Kasparov Theorem**

- **Abelian Groups**
  - **Amenable Groups**
    - **Exact Groups**
      - **a-T-menable Groups**
        - **BCC**

**The Higson-Kasparov Theorem (2001)**

Any second countable a-T-menable group $G$ satisfies *BCC*

Their proof involves (infinite-dimensional) functional analysis; (no classical differential geometry or representation theory).
**Setting**

- $\mathcal{H}$: a separable (real) Hilbert space.
- $G$: an a-T-menable group acting properly on $\mathcal{H}$. 
**Dual Dirac Method** \( (1_G = \gamma_G) \)

- \( KK^G \): Kasparov’s equivariant \( KK \)-theory

### The Dual Dirac Method

The following is sufficient to prove BCC for \( G \):

- Find a **proper** \( G \)-C*-algebra \( A(\mathcal{H}) \);
- Find an element \( b \in KK^G_* (\mathbb{C}, A(\mathcal{H})) \) (Bott element);
- Find an element \( d \in KK^G_* (A(\mathcal{H}), \mathbb{C}) \) (Dirac element);

such that

- The product \( b \otimes_{A(\mathcal{H})} d = 1_G \) in \( KK^G(\mathbb{C}, \mathbb{C}) \).

A \( G \)-C*-algebra \( A \) is **proper** if there is a proper \( G \)-space \( X \) and a nondegenerate equivariant \(*\)-homomorphism from \( C_0(X) \) to the center of the multiplier algebra \( M(A) \).
Strategy

There is a finite dimensional version of what we want to do:

When Hilbert space is finite dimensional ($\mathcal{H} = \mathbb{R}^n$)

- $C_\tau(\mathbb{R}^n) = \Gamma_0(\mathbb{R}^n, \text{Cliff}(\mathbb{R}^n))$ (proper algebra);
- $b_{\mathbb{R}^n} \in KK_0^G(\mathbb{C}, C_\tau(\mathbb{R}^n))$ (Bott element);
- $d_{\mathbb{R}^n} \in KK_0^G(C_\tau(\mathbb{R}^n), \mathbb{C})$ (Dirac element);
- $b_{\mathbb{R}^n} \otimes_{C_\tau(\mathbb{R}^n)} d_{\mathbb{R}^n} = [L^2(\mathbb{R}^n, \Lambda^*_\mathbb{C}(\mathbb{R}^n)), B_{\mathbb{R}^n}]$;
- $[L^2(\mathbb{R}^n, \Lambda^*_\mathbb{C}(\mathbb{R}^n)), B_{\mathbb{R}^n}] = 1_G$ in $KK^G(\mathbb{C}, \mathbb{C})$.

Here, $B_{\mathbb{R}^n}$ is a Bott-Dirac operator of $\mathbb{R}^n$.

$n=1$

- $c(e_1) (\bar{c}(e_1))$: (skew-) s.a. Clifford mult. by $e_1 \in \mathbb{R}$;
- $b_{\mathbb{R}} = [C_\tau(\mathbb{R}), c(e_1)x] \in KK_0^G(\mathbb{C}, C_\tau(\mathbb{R}))$;
- $d_{\mathbb{R}} = [C_\tau(\mathbb{R}) \bowtie L^2(\mathbb{R}, \Lambda^*_\mathbb{C}(\mathbb{R})), \bar{c}(e_1) \frac{d}{dx}] \in KK_0^G(C_\tau(\mathbb{R}), \mathbb{C})$;
- $b_{\mathbb{R}} \otimes_{C_\tau(\mathbb{R})} d_{\mathbb{R}} = [L^2(\mathbb{R}, \Lambda^*_\mathbb{C}(\mathbb{R})), B_{\mathbb{R}} = c(e_1)x + \bar{c}(e_1) \frac{d}{dx}]$. 
The proof of the Higson-Kasparov Theorem is nothing but to make sense of the following “limits”:

- \( A(\mathcal{H}) := \text{“} \lim C_t(\mathbb{R}^n) \text{”} \) (proper algebra);
- \( b := \text{“} \lim b_{\mathbb{R}^n} \in KK^G_1(\mathbb{C}, A(\mathcal{H})) \text{”} \) (Bott element);
- \( d := \text{“} \lim d_{\mathbb{R}^n} \in KK^G_1(A(\mathcal{H}), \mathbb{C}) \text{”} \) (Dirac element);
- \( \text{“} \lim[L^2(\mathbb{R}^n, \Lambda^\mathbb{C}_*(\mathbb{R}^n)), B_{\mathbb{R}^n}] = 1_G \in KK^G(\mathbb{C}, \mathbb{C}) \text{”} \).

An interesting analysis is used when we deal with the “limit” of the cycles \([L^2(\mathbb{R}^n, \Lambda^\mathbb{C}_*(\mathbb{R}^n)), B_{\mathbb{R}^n}]\).
(Non-commutative functional calculus).
The Bott-Dirac Operator $B_{\mathbb{R}^n}$ represents $1_G$ in $KK^G(\mathbb{C}, \mathbb{C})$.

$$B_{\mathbb{R}} := c(e_1)x + \tilde{c}(e_1) \frac{d}{dx} = \begin{pmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{pmatrix}$$

- $B_{\mathbb{R}}$ is an odd unbounded operator on $L^2(\mathbb{R}, \Lambda^*_C(\mathbb{R}))$;
- It is defined on the Schwartz space $s(\mathbb{R}, \Lambda^*_C(\mathbb{R}))$;
- It is selfadjoint and diagonalizable;
- It has compact resolvent; and $\text{Ker}B_{\mathbb{R}} = \text{span}\{e^{-\frac{||x||^2}{2}}\}$.

$$B_{\mathbb{R}} = \begin{pmatrix}
0 & 0 & 0 & \cdots \\
0 & \sqrt{2} & 0 & \cdots \\
0 & 0 & \sqrt{4} & \cdots \\
0 & 0 & 0 & \sqrt{6} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$
Bott-Dirac Operator of $\mathbb{R}^n$

$$B_{\mathbb{R}^n} := \sum_{j=1}^{n} c(e_j)x_j + \bar{c}(e_j)\frac{\partial}{\partial x_j}$$

$$= B_{\mathbb{R}^n} \hat{\otimes} 1 \hat{\otimes} \cdots \hat{\otimes} 1 + 1 \hat{\otimes} B_{\mathbb{R}^n} \hat{\otimes} 1 \hat{\otimes} \cdots \hat{\otimes} 1 + \cdots + 1 \hat{\otimes} \cdots \hat{\otimes} 1 \hat{\otimes} B_{\mathbb{R}^n}$$

$$L^2(\mathbb{R}^n, \Lambda^c_*(\mathbb{R}^n)) = L^2(\mathbb{R}, \Lambda^c_*(\mathbb{R})) \hat{\otimes} \cdots \hat{\otimes} L^2(\mathbb{R}, \Lambda^c_*(\mathbb{R}))$$

- $B_{\mathbb{R}^n}$ is an odd unbounded operator on $L^2(\mathbb{R}^n, \Lambda^c_*(\mathbb{R}^n))$;
- It is defined on the Schwartz space $s(\mathbb{R}^n, \Lambda^c_*(\mathbb{R})^n)$;
- It is independent of the choice of a basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$;
- It is selfadjoint and diagonalizable;
- It has compact resolvent; and $\text{Ker}B_{\mathbb{R}^n} = \text{span}\{e^{-\frac{||x||^2}{2}}\}$;
- Note: the eigenspace of $B_{\mathbb{R}^n}^2$ for $\lambda = 2$ has dimension $2n$. 
We want to define Bott-Dirac Operator $B_{\mathcal{H}}$ of an infinite dimensional Hilbert space $\mathcal{H}$ as an inductive limit of $B_{\mathbb{R}^n}$.

There is a natural construction of such an inductive limit.
Bott-Dirac Operator of $\mathcal{H}$

- $\mathcal{H}$: a separable (real) ($G$-)Hilbert space.
- For each finite dimensional subspace $V$ of $\mathcal{H}$,
  $$H(V) := L^2(V, \wedge^* (V))$$
  $$s(V) := s(V, \wedge^* (V))$$ (Schwartz space);
  $B_V$: Bott-Dirac Operator of $V$
- For an inclusion of subspaces $V \subset V' = V \oplus W$,
  $$H(V) \to H(V') = H(V) \hat{\otimes} H(W): \xi \mapsto \xi \hat{\otimes} e^{-\frac{||w||^2}{2}}$$
  $$H(\mathcal{H}) := \lim_V H(V): \text{naturally } G\text{-Hilbert space.}$$
- $$B_\mathcal{H} := \lim_V B_V$$
- $B_\mathcal{H}$ is defined on $s(\mathcal{H}) := \text{alg-lim}_V s(V)$;
- It is a well-defined odd unbounded operator on $H(\mathcal{H})$. 
Bott-Dirac Operator of $\mathcal{H}$

$$B_\mathcal{H} := \lim_{V} B_V$$

For $\xi \in s(V) \subset s(\mathcal{H})$, $B_\mathcal{H}(\xi) := B_V \xi \in s(V) \subset s(\mathcal{H})$.

It is well-defined because the following diagram commutes for $V \subset V \oplus W \subset \mathcal{H}$:

$$
\begin{array}{ccc}
s(V) & \longrightarrow & s(V \oplus W) \\
\downarrow B_V & & \downarrow B_{V \oplus W} \\
s(V) & \longrightarrow & s(V \oplus W)
\end{array}
$$

To see this, one may write $B_{V \oplus W} = B_V + B_W$. For $\xi \in s(V)$, $B_{V \oplus W}(\xi \hat{\otimes} e^{-\frac{||w||^2}{2}}) = B_V \xi \hat{\otimes} e^{-\frac{||w||^2}{2}} + \xi \hat{\otimes} B_W e^{-\frac{||w||^2}{2}} = B_V \xi \hat{\otimes} e^{-\frac{||w||^2}{2}}$. 
Bott-Dirac Operator of $\mathcal{H}$

$$B_{\mathcal{H}} := \lim_{V} B_{V}$$

- $B_{\mathcal{H}}$ is an odd unbounded operator on $H(\mathcal{H}) := \lim_{V} H(V)$;
- It is selfadjoint and diagonalizable;
- It is $G$-equivariant if $G$ acts on $\mathcal{H}$ linearly;
- $\text{Ker} B_{\mathcal{H}} = \text{span}\{ e^{-\frac{\|x\|^2}{2}} \}$;
- It does not have compact resolvent:
  - the eigenspace of $B_{\mathcal{H}}^{2}$ for $\lambda = 2$ has infinite dimension.

$[H(\mathcal{H}), B_{\mathcal{H}}]$ doesn’t define an element in $KK^{G}(\mathbb{C}, \mathbb{C})$. 
Quick Solution

After fixing some basis \( \{e_j\}_{j=1}^\infty \) of \( \mathcal{H} \), we may write \( B_\mathcal{H} \) as:

\[
B_\mathcal{H} = \sum_{j=1}^\infty c(e_j)x_j + \bar{c}(e_j) \frac{\partial}{\partial x_j}
\]

\[
= B_R \hat{\otimes} 1 \hat{\otimes} \cdots + 1 \hat{\otimes} B_R \hat{\otimes} 1 \hat{\otimes} \cdots + \cdots
\]

\[
H(\mathcal{H}) = L^2(\mathbb{R}, \Lambda^C(\mathbb{R})) \hat{\otimes} L^2(\mathbb{R}, \Lambda^*_C(\mathbb{R})) \hat{\otimes} \cdots
\]

A quick solution for the non-compact resolvent issue is:

**Proposition**

Fix some basis as above. For any unbounded increasing sequence \( (n_k) \) of positive numbers,

\[
\tilde{B}_\mathcal{H} := n_1 B_R \hat{\otimes} 1 \hat{\otimes} \cdots + n_2 1 \hat{\otimes} B_R \hat{\otimes} 1 \hat{\otimes} \cdots + n_3 1 \hat{\otimes} 1 \hat{\otimes} B_R \hat{\otimes} \cdots + \cdots
\]

defines an unbounded, diagonalizable selfadjoint operator on \( H(\mathcal{H}) \) having compact resolvent with \( \text{Ker} B_\mathcal{H} = \text{span}\{e^{-\|x\|^2/2}\} \).
Non-Commutative Functional Calculus

Non-commutative functional calculus is a more systematic way to do such perturbation of $B_\mathcal{H}$.

**Non-Commutative Functional Calculus (Higson, Kasparov)**

- $h$: any symmetric, densely defined operator on $\mathcal{H}$;
- $\mathcal{H}_h$: the domain of $h$;
- $h(B_\mathcal{H}) := \sum_{j=1}^{\infty} c(he_j)x_j + \bar{c}(he_j)\frac{\partial}{\partial x_j}$;
- $h(B_\mathcal{H})$ is defined on $\text{alg- lim}_{V \subseteq \mathcal{H}_h} s(V)$;
- It is a well-defined odd symmetric operator on $H(\mathcal{H})$;
- It is independent of the choice of a basis $\{e_j\}_{j=1}^{\infty}$ of $\mathcal{H}_h$;
- When $h$ is diagonalizable, so is $h(B_\mathcal{H})$;
- When $h$ has compact resolvent, so is $h(B_\mathcal{H})$;
- The assignment $h \mapsto h(B_\mathcal{H})$ is $\mathbb{R}$-linear.
Let's take a closer look at the definition:
Consider for any densely defined operator $h$ on $\mathcal{H}$,

- $V_n := \text{span}\{ e_j | j = 1, \cdots, n \} \subset \mathcal{H}_h$
- $h(B_{V_n}) := \sum_{j=1}^{n} c(he_j)x_j + \bar{c}(he_j)\frac{\partial}{\partial x_j}$

Can we define $h(B_{\mathcal{H}}) := \lim h(B_{V_n})$?

For $V_n \subset V_n' \subset V_n'' \subset \mathcal{H}_h$ and $V_n'' + hV_n'' \subset W$:

we may hope that the following diagram commutes:

$$
\begin{array}{ccc}
\text{s}(V_n, \Lambda^C_*(V_n)) & \longrightarrow & \text{s}(V_n', \Lambda^C_*(V_n')) \\
\downarrow h(B_{V_n'}) & & \downarrow h(B_{V_n''}) \\
\text{s}(V_n', \Lambda^C_*(W)) & \longrightarrow & \text{s}(V_n'', \Lambda^C_*(W))
\end{array}
$$
Non-commutative Functional Calculus

In the paper by Higson and Kasparov, it was (implicitly) claimed that this diagram commutes:

\[
s(V_n, \Lambda^C_*(V_n)) \longrightarrow s(V_n', \Lambda^C_*(V_n')) \longrightarrow s(V_n'', \Lambda^C_*(V_n''))
\]

\[
\downarrow \quad h(B_{V_n'}) \quad \quad \quad \downarrow \quad h(B_{V_n''})
\]

\[
s(V_n', \Lambda^C_*(W)) \longrightarrow s(V_n'', \Lambda^C_*(W))
\]

However, in general, this diagram does not commute.

**Theorem (Fixed non-commutative functional calculus) (N.)**

- The diagram asymptotically commutes iff \( h^* \) is defined on \( V_n \).
- The following formula defines \( h(B_{\mathcal{H}}) \) unambiguously for any \( h \) whose adjoint \( h^* \) is defined on \( \mathcal{H}_h \):

\[
\xi \in s(V), \quad h(B_{\mathcal{H}})(\xi) := \lim_{W \subset \mathcal{H}_h} h(B_{V \oplus W})(\xi \otimes e^{-\frac{||w||^2}{2}})
\]
Example 1: consider if $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on span\{e_1, e_2\};

This is not commutative due to the following nonzero term:

$$(c(h e_2)x_2 + \bar{c}(h e_2)\frac{\partial}{\partial x_2})(\xi \hat{\otimes} e^{-\frac{x_2^2}{2}})$$

$$= \text{int}(e_1)\xi \hat{\otimes} 2x_2 e^{-\frac{x_2^2}{2}} \ (\text{for } \xi \in s(V_1)).$$
Example 2: consider:

\[
h = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 2 & 0 & 0 & \cdots \\
0 & 0 & 3 & 0 & \cdots \\
0 & 0 & 0 & 4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} + \begin{pmatrix}
1 \\
\frac{1}{2} \\
\frac{1}{3} \\
\frac{1}{4} \\
\vdots
\end{pmatrix} (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots);
\]

\[
s\left(V_1, \Lambda^C_*(V_1)\right) \rightarrow s\left(V_n', \Lambda^C_*(V_n')\right) \rightarrow s\left(V_n'', \Lambda^C_*(V_n'')\right)
\]

\[
\downarrow h(B_{V_n'}) \quad \downarrow h(B_{V_n''})
\]

\[
s\left(V_n', \Lambda^C_*(W)\right) \rightarrow s\left(V_n'', \Lambda^C_*(W)\right)
\]

This is not commutative no matter how large we take \(V_n'\). Nonetheless, it asymptotically commutes.
So what is \( \lim[L^2(\mathbb{R}^n, \Lambda^*_C(\mathbb{R}^n)), B_{\mathbb{R}^n}] \)?

With suitable \( h \), one can guarantee “asymptotic equivariance” for a family \( \{(1 + th)(B_{\mathcal{H}})\}_{t > 0} \).

“Via asymptotic morphisms”, one may define “\([H(\mathcal{H}), (1 + th)(B_{\mathcal{H}})] = 1_G \in KK^G(\mathbb{C}, \mathbb{C})\)."
$A(\mathcal{H})$: $C^*$-algebra of Hilbert space $\mathcal{H}$

The following “limits” can be defined without so much trouble:

- $A(\mathcal{H}) := \text{“lim } C_\tau(\mathbb{R}^n)\text{” (proper algebra)}$;
- $b := \text{“lim } b_{\mathbb{R}^n}\text{” (Bott element)}$;
- $S = C_0(\mathbb{R})$: a (graded) $C^*$-algebra.

$A(\mathcal{H}) := \lim_{\mathcal{V}} S \hat{\otimes} C_\tau(V)$ is defined as follows:
- For an inclusion of subspaces $V \subset V' = V \oplus W$,

  $S \hat{\otimes} C_\tau(V) \to S \hat{\otimes} C_\tau(V') \cong S \hat{\otimes} C_\tau(W) \hat{\otimes} C_\tau(V)$:

It’s given by the graded tensor product of $*$-homomorphisms:

- $S \to S \hat{\otimes} C_\tau(W) : f \mapsto f(x \hat{\otimes} 1 + 1 \hat{\otimes} c_W)$
- $C_\tau(V) \to C_\tau(V) : \text{the identity on } C_\tau(V)$
Explanation for $S \rightarrow S\hat{\otimes}C_{\tau}(W) : f \mapsto f(x\hat{\otimes}1 + 1\hat{\otimes}c_{W})$:

- $x$ is an (odd) unbounded multiplier on $S$: multiplication by $x$ at $x$ in $\mathbb{R}$;
- $c_{W}$ is an (odd) unbounded multiplier on $C_{\tau}(W)$: Clifford multiplication $c(w)$ at $w$ in $W$;

$x\hat{\otimes}1 + 1\hat{\otimes}c_{W}$ is an (odd) unbounded multiplier on $S\hat{\otimes}C_{\tau}(W)$.

We have a functional calculus $f \mapsto f(x\hat{\otimes}1 + 1\hat{\otimes}c_{W})$.

For example:

$$e^{-x^2} \mapsto e^{-x^2}\hat{\otimes}e^{-\|w\|^2};$$

$$xe^{-x^2} \mapsto xe^{-x^2}\hat{\otimes}e^{-\|w\|^2} + e^{-x^2}\hat{\otimes}c_{W}e^{-\|w\|^2}.$$
$A(\mathcal{H})$: $C^*$-algebra of Hilbert space $\mathcal{H}$

- When a group $G$ acts on $\mathcal{H}$ affine isometrically, the $C^*$-algebra $A(\mathcal{H})$ is naturally a $G$-$C^*$-algebra.

- If moreover, $G$ acts on $\mathcal{H}$ (metrically) properly, the $C^*$-algebra $A(\mathcal{H})$ is a proper $G$-$C^*$-algebra.

- Indeed, the center of $A(\mathcal{H})$ is $C_0([0, \infty) \times \mathcal{H})$

- $\lim_{W} x \hat{\otimes} 1 + 1 \hat{\otimes} c_W$ defines an element $b$ in $KK_1^G(\mathbb{C}, A(\mathcal{H}))$. 
“\( \lim d_{\mathbb{R}^n} \)” and Spectral Dual-Dirac

- It is somewhat technical to construct the Dirac element \( d := \lim d_{\mathbb{R}^n} \) with bare hands in \( KK^G \).

What we can simply have is the following “Spectral Dual Dirac”

- \( A \rightarrow B \) denotes an asymptotic morphism from \( A \) to \( B \)
- \( A(\mathcal{H}) \): the C*-algebra of Hilbert space \( \mathcal{H} \) (proper algebra)
- \( \beta : S \rightarrow A(\mathcal{H}) \) (“Bott element”);
- \( \alpha : A(\mathcal{H}) \rightarrow S \hat{\otimes} K(H(\mathcal{H})) \) (“Dirac element”);
- The composition \( \alpha \circ \beta : S \rightarrow S \hat{\otimes} K(H(\mathcal{H})) \) is homotopic to \( \text{id}_S : S \rightarrow S \) in a suitable sense.

Indeed, the idea of Higson and Kasparov was to translate (lift) everything into the language of \( KK \)-theory:
Higson and Kasparov lifted this Spectral Dual-Dirac to $KK$ in the following way:

- We already have a Bott element $b \in KK_1^G(\mathbb{C}, A(\mathcal{H}))$ which “corresponds to” the “Bott element” $\beta : S \to A(\mathcal{H})$;
- We can construct an extension of $G$-$C^*$-algebras

\[
\begin{array}{cccc}
0 & \to & J & \to & B & \to & A(\mathcal{H}) & \to & 0
\end{array}
\]

which “corresponds to” the “Dirac element” $\alpha : A(\mathcal{H}) \to S \hat{\otimes} K(H(\mathcal{H}))$;
- Although this extension may not be equivariant semi-split, we can show there is an element $d$ in $KK_1^G(A(\mathcal{H}), J)$ which “corresponds to” this extension;
- One can compute the Kasparov product $b \otimes_{A(\mathcal{H})} d \cong 1_G$. 

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Spectral Dual-Dirac lifts

The following simplifies what actually happened in the proof:

**Theorem (Spectral Dual-Dirac lifts) (N.)**

For any $G$, suppose we have the following “Spectral Dual-Dirac”:

- $H$: a (complex, graded) $G$-Hilbert space;
- $A$: a proper nuclear $C^*$-algebra;
- $\beta : S \to A$ (“Bott element”);
- $\alpha : A \to S \hat{\otimes} K(H)$ (“Dirac element”);
- The composition $\alpha \circ \beta : S \to S \hat{\otimes} K(H)$
  is “homotopic” to $\text{id}_S : S \to S$.

Then, this lifts to Dual-Dirac ($\gamma_G = 1_G$) in $KK$-theory if there is a $b \in KK_1^G(\mathbb{C}, A)$ which “corresponds to” $\beta$.

i.e. The Spectral Dual-Dirac lifts if the Bott element lifts.

Note, in our setting, $A(\mathcal{H})$ is nuclear and the Bott element indeed lifts.
This concludes the proof of the Higson-Kasparov Theorem.
Summary

- The proof of the Higson-Kasparov Theorem is nothing but precisely making sense of a limit of Dual-Dirac method for finite dimensional case.

- A noncommutative functional calculus is needed since the naive Bott-Dirac operator in infinite dimensions does not have compact resolvent. Fixed version of this not only gives a precise formula which was not mentioned in the work by Higson and Kasparov but also shows one can apply it with respect to any bounded operators.

- The lifting of the Dual-Dirac method from E-theory to KK-theory can be simplified.
THANK YOU VERY MUCH!!
Some References

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