Einstein-Hilbert action on Connes-Landi noncommutative manifolds

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Analysis, Noncommutative Geometry, Operator Algebras Workshop
June 2017

Motivations and History

Motivation:

- Explore notions of metric and curvature for certain class of noncommutative manifolds.
- Quantum gravity

On noncommutative two torus:

- Connes's pseudo differential calculus 80's. The computation was initiated at the same time;
- Gauss-Bonnet theorem (2011): Connes-Tretkoff, Khalkhali-Fathizadeh;
- Conformal geometry (2014): Connes-Moscovici, Khalkhali-Fathizadeh, Lesch, Moscovici-Lesch and others.

This talk is based on my work arXiv 1510.04668 (to appear in JNCG), arXiv 1611.08933.

Connes-Landi noncommutative manifolds $C^\infty(M_\theta)$

Let M be a Riemannian manifold such that $\mathbb{T}^n \subset \mathrm{Diff}(M)$.

- $C^\infty(M)$ becomes a $\mathbb{T}^n$-module through the dual action: for $t \in \mathbb{T}^n$, $U_t : C^\infty(M) \to C^\infty(M)$:

$$U_t(f)(x) = f(t^{-1} \cdot x), \ \ \forall f \in C^\infty(M).$$

- Isotypic decomposition: $\forall f \in C^\infty(M)$,

$$f = \sum_{r \in \mathbb{Z}^n} f_r, \ \ \text{with } f_r = \int_{\mathbb{T}^n} e^{-2\pi i r \cdot t} U_t(f) dt.$$

In particular,

$$fh = \sum_{r,l} f_r h_l.$$

θ-deformation (M. Rieffel)

- For any $n \times n$ skew symmetric matrix θ, denote by bicharacter on $\mathbb{Z} \times \mathbb{Z}$: $\chi_\theta(r, l) = e^{2\pi i \langle \theta r, l \rangle}$.
- One can construct a new family of multiplication for $C^\infty(M)$ by twisting the convolution: let $f = \sum_r f_r$ and $g = \sum_l g_l$,

$$f \times_\theta h = \sum_{r,l} \chi_\theta(r, l) f_r h_l.$$

- The new algebra $C^\infty(M_\theta) \triangleq (C^\infty(M), \times_\theta)$ represents the "smooth structure" of M_θ.
- Natural vector bundles over M_θ ($C^\infty(M_\theta)$-bimodules): f smooth function, X vector field, ω one form: $f \times_\theta X$, $\omega \times_\theta f$, $X \cdot_\theta \omega$ etc.
- Deforming linear operators $P : C^\infty(M) \to C^\infty(M)$: $\pi^\theta(P)(f) \triangleq P \times_\theta f$.

Quantum mixed with Riemannian

Quantum side

In general, the $\times_\theta$-multiplication is highly nonlocal. For example, the evaluation of $(f|_U) \times_\theta (h|_U)$ at some point $x \in U$ depends on U in such a obscure way that the value has no compatibility when U is shrinking around x.

Riemannian side

If one of h and f is $\mathbb{T}^n$-invariant, then $f \times_\theta h = fh$. In particular,

- For a $\mathbb{T}^n$-invariant metric g, the associated connection and spinor Dirac are invariant under the deformation:

$$\pi^\theta(\nabla) = \nabla, \ \ \pi^\theta(\not{D}) = \not{D}.$$

- Leibniz property holds when computing $\nabla(X \cdot_\theta \omega)$.
- The spectral triple $(C^\infty(M_\theta), L^2(\$), \not{D})$ satisfies Connes's axioms of spin geometry.

Scalar Curvature functional (Spectral Geometry)

For a given metric g with the associated spinor Dirac $\not{D}$, the local geometry is encoded in the heat kernel asymptotic:

$$\mathrm{Tr}(fe^{-t\not{D}^2}) \backsim_{t\searrow 0} \sum_{j=0}^{\infty} V_j(f,\not{D}^2)t^{\frac{j-m}{2}} \quad m = \dim M.$$

- Coefficients are functionals: $f \in C^\infty(M) \to V_j(f,\not{D}^2 a) \in \mathbb{C}$.
- To "localize" the functionals, introduce $\varphi_0(\cdot) = \int_M \mathrm{Tr}_x(\cdot)dg$
- Functional density $V_j(x) \in C^\infty(M)$ is defined by the property

$$V_j(f,\not{D}^2) = \varphi_0\left(f(x)V_j(x)\right)$$

- In general, $V_j(x)$ is a polynomial whose arguments are components of the curvature tensor and the derivatives.

- What we need

$$\boxed{V_2(x) = -\frac{1}{12}\mathcal{S}_g.}$$

  where $\mathcal{S}_g$ is the scalar curvature.
- $V_2(\cdot, \not{D}^2)$: scalar curvature functional.

The definition is intrinsic with respect to the spectral data:

- Define

$$⨍ P = \mathrm{Res}_{z=0} \operatorname{Tr}\left(P\not{D}^{-z}\right).$$

- The functional φ_0 is proportional to

$$⨍ (\cdot)\not{D}^{-m}, \quad m = \dim M.$$

- The scalar curvature functional is proportional to

$$⨍ (\cdot)\not{D}^{-m+2}, \quad m = \dim M.$$

Symmetry Breaking

$\mathbb{T}^n$-invariant metrics behave like the Riemannian version

- The spectral triple $(C^\infty(M_\theta), L^2(\not{S}), \not{D})$ satisfies Connes's axioms of spin geometry.
- Leibniz property holds when computing $\nabla(X \cdot_\theta \omega)$.

The rest of the metrics seems intangible, because there is no straightforward calculus.

- Widom's work tells us that any connection ∇ will lead to a pseudo differential calculus.
- My work started with the observation: if $\nabla = \nabla^g$ for some $\mathbb{T}^n$-invariant metric g, such calculus can be deformed to study the spectral geometry of M_θ.

Quantizing metrics

Let $g' = e^{-2h}g$ for a real-valued $h \in C^\infty(M)$. We would like to compare the two spinor Dirac $\not{D}^{g'}$ and $\not{D}^{g}$. There is a canonical and unitary identification between the L^2-spinor sections

$$\tilde{\beta}_g^{-g'} : L^2(\$^g) \to L^2(\$^{g'}).$$

Proposition

The g'-spinor Dirac $\not{D}^{g'}$ is unitary equivalent to

$$\boxed{\tilde{\beta}^g_{-g'} \not{D}^{g'} \tilde{\beta}_g^{-g'} = e^{h/2} \not{D}^g e^{h/2}.}$$

Conformal Geometry

Let us fix a $\mathbb{T}^n$-invariant metric g as a background metric. The conformal class $[g]$ can be parametrized by (g, h) with $h = h^* \in C^\infty(M)$.

(M, g)	$(C^\infty(M_\theta), L^2(\not{S}), \not{D})$
e^{-2h} with $h \in C^\infty(M_\theta)$ real-valued	e^{-2h} with $h = h^* \in C^\infty(M_\theta)$ self-adjoint
$g' = e^{-2h} g$	$D_h = e^{h/2} \not{D} e^{h/2}$
Scalar curvature for g': $\mathcal{S}_{g'}$	$R_{D_h} \in C^\infty(M_\theta)$: functional density of $V_2(\cdot, D_h^2)$

Table: Conformal change of metric and the associated scalar curvature in the noncommutative setting.

Modular scalar curvature

- Weyl factor (conformal factor), $k = e^h$ with $h = h^* \in C^\infty(M_\theta)$.
- Modular derivation (noncommutative differential): $\mathbf{x}_h = -\mathbf{ad}_h = [\cdot, h]$. Modular operator: $\mathbf{y}_h = \exp \mathbf{x}_h = k^{-1}(\cdot)k$.

Theorem (Liu, 16)

Let $\dim M = 4$. *For* $g' = e^{-2h} g$ *or* $\not{D} \mapsto D_h = e^{h/2} \not{D} e^{h/2}$, *upto a constant factor, the modular scalar curvature defined in Table* (1) *is given by*

$$R_{g'} = e^{(-m+2)h} \left(K(\mathbf{x}_h)(\nabla^2 h) + H(\mathbf{x}_h^{(1)}, \mathbf{x}_h^{(2)})(\nabla h \nabla h) \right) \cdot g^{-1} + c\mathcal{S}_g$$

Notations: write H as a Fourier transform: $H = \mathcal{F}\beta$,

$$H(\mathbf{x}_h^{(1)}, \mathbf{x}_h^{(2)})(\psi \cdot \varphi) = \int_{\mathbb{R}^2} \beta(u, v)(\mathbf{y}_h)^{-iu}(\psi)(\mathbf{y}_h)^{-iv}(\varphi) du dv.$$

The one-variable function K

Assume $\dim M = 4$.

$$K(u) = -\frac{1}{2} e^u \frac{\sinh(u/2)}{(u/2)},$$

which shows striking similarity to the AS local index formula.

- The $\hat{A}$-genus part comes from the fact that the deformation start on the tangent space of the metrics:

$$\nabla e^h = f_1(\mathrm{ad}_h)(\nabla h) = e^h \frac{e^{\mathrm{ad}_h} - 1}{\mathrm{ad}_h}(\nabla h).$$

- In local index formula, the $\hat{A}$-genus naturally through the Jacobian of the exponential map of a Lie group G: in exponential coordinates $d \exp h = j_{\mathfrak{g}}(h) dh$, where $\mathfrak{g}$ is the Lie algebra, dh is a Haar measure and

$$j_{\mathfrak{g}} = \mathrm{det}_{\mathfrak{g}}\left(-f_1(\mathrm{ad}_h)\right) = \mathrm{det}_{\mathfrak{g}}\left(-\frac{e^{\mathrm{ad}_h} - 1}{\mathrm{ad}_h}\right).$$

Einstein Hilbert action

- Recall
$$R_{g'} = e^{(-m+2)h}\left(K(\mathbf{x}_h)(\nabla^2 h) + H(\mathbf{x}_h^{(1)}, \mathbf{x}_h^{(2)})(\nabla h \nabla h)\right) \cdot g^{-1} + c\mathcal{S}_g$$
- For each $g' = e^{-2h}g$. We have the scalar curvature functional (on $C^\infty(M)$):
$$f \mapsto V_2(f, D_h^2) = \varphi_0(fR_{g'}).$$
- The noncommutative analogue of the Einstein Hilbert action is given by
$$F_{\mathrm{EH}}(h) \triangleq V_2(1, D_h^2)$$
which has a local expression in dim four:
$$F_{\mathrm{EH}}(h) = \varphi_0\left(R_{g'}\right) = \varphi_0\left(e^{-2h}[T(\mathbf{x}_h)(\nabla h)\cdot(\nabla h)]g^{-1} - \frac{1}{12}e^{-2h}\mathcal{S}_g\right).$$

Local curvature functions in dim 4

- Modular scalar curvature:

$$K(2u) = -\frac{2e^{-\frac{u}{2}} \sinh\left(\frac{u}{4}\right)}{u},$$

$$H(2u, 2v) = \frac{4e^{\frac{1}{4}(-3)(u+v)} \left(u\left(-e^{v/2}\right) + \left(e^{u/2} - 1\right) e^{v/2} v + u\right)}{uv(u+v)},$$

$$c = K(0)/6.$$

- Einstein Hilbert action F_{EH}:

$$T(2u) = 2K(0) f_1(-2u) + H(2u, -2u)$$
$$= \frac{e^{-u} - 1}{u} - \frac{-2u - 4e^{-\frac{u}{2}} + 4}{u^2}.$$

- Riemannian case is the commutative limit

$$K(0) = -\frac{1}{2}, H(0,0) = \frac{1}{2}$$

$$-\frac{1}{12} e^{-mh} \mathcal{S}_{g'} = R_{g'}\Big|_{\mathbf{x}_h = 0}$$

Variational Calculus

For any $a = a^* \in C^\infty(M_\theta)$, for $\varepsilon > 0$, denote $h_\varepsilon = h + \varepsilon a$. Set $\delta_a = d/d\varepsilon|_{\varepsilon=0}$. The gradient at h: $\operatorname{grad}_h F_{\mathrm{EH}}$ is define by the equation: for any $a = a^* \in C^\infty(M_\theta)$,

$$\delta_a F_{\mathrm{EH}}(h) = \varphi_0(a \operatorname{grad}_h F_{\mathrm{EH}}).$$

By differentiating $\delta_a \operatorname{Tr}(e^{tD_h^2})$, we obtain:

Proposition

Let $m = \dim M$*,*

$$\delta_a V_2(1, D_\varepsilon^2) = \frac{2-m}{2} V_2\Big(\int_{-1}^{1} e^{uh/2} a e^{-uh/2} du, D_h^2\Big).$$

Proposition

Let $m = \dim M$ *and* $j_0 = (m-2)/2$. *The gradient* $\operatorname{grad}_h F_{\mathrm{EH}}$ *is given by*

$$\operatorname{grad}_h F_{\mathrm{EH}} = e^{-2j_0 h}\left(K_{\mathrm{EH}}(\mathbf{x}_h)(\nabla^2 h) + H_{\mathrm{EH}}(\mathbf{x}_h^{(1)}, \mathbf{x}_h^{(2)})(\nabla h \nabla h)\right) g^{-1}$$
$$+ (-2j_0)c(m)e^{-2j_0 h}\mathcal{S}_\Delta,$$

where

$$K_{\mathrm{EH}}(2s) = -8j_0 \frac{\sinh(s/4)}{s} K(2s),$$
$$H_{\mathrm{EH}}(2s, 2t) = -8j_0 \frac{\sinh((s+t)/4)}{s+t} H(2s, 2t)$$

On the other hand, we can compute the gradient $\mathrm{grad}_h F_{\mathrm{EH}}$ directly from the local expression

$$\delta_a \left[\varphi_0(R_{g'})\right] = \delta_a \left[\varphi_0\left(e^{-2h}[T(\mathbf{x}_h)(\nabla h)\cdot(\nabla h)]g^{-1} - \frac{1}{12}e^{-2h}\mathcal{S}_g\right)\right].$$

Theorem (Liu, 16)

The internal relations are given by

$$K_{\mathrm{EH}}(s) = -(T+\tilde{T})(s),$$
$$H_{\mathrm{EH}}(s,t) = (L+M-P-Q)(s,t),$$

where $\tilde{T}(s) = T(-s)e^{-s}$ and functions L, M, P and Q are given in the next page.

Some two-variable functions

- Recall: $f_1(u) = (e^u - 1)/u$ and $\tilde{T} = T(-s)e^{-s}$.
- Divided difference

$$[u, v]T \triangleq \frac{T(u) - T(v)}{u - v}.$$

- Then

$$\begin{aligned}
L(s,t) &= -2T(s)f_1(-(s+t)), \\
M(s,t) &= 2e^{-(s+t)}\left(e^t[t,-s](T) - [s,-t](T)\right), \\
P(s,t) &= -2f_1(-s)T(t) - 2[s+t,t](T) + 2[s+t,s](T), \\
Q(s,t) &= -2f_1(-s)\tilde{T}(t) - 2[s+t,t](\tilde{T}) + 2[s+t,s](\tilde{T})
\end{aligned}$$

- (Lesch): the appearance of $-2[s+t,t](T) + 2[s+t,s](T)$ is due to the commutator of commutative and noncommutative derivations:

$$[\nabla, T(\mathbf{x}_h)](\psi).$$

Independence of the background metric

- Let g' be the noncommutative metric in question, g_0 and g_1 are two background metrics ($\mathbb{T}^n$-invariant).
- $g' = e^{-2h} g_1 = e^{-2(h+a)} g_0$ with $a = a^* \in C^\infty(M_\theta)$ commutative and $g_1 = e^{-2a} g_0$.
- We can quantize g' in g_1 and g_0:

$$g' \mapsto e^{h/2} \not{D}^{g_1} e^{h/2} \quad \text{or} \quad g' \mapsto e^{(h+a)/2} \not{D}^{g_0} e^{(h+a)/2}.$$

- Since $\not{D}^{g_1}$ is unitary equivalent to $e^{a/2} \not{D}^{g_0} e^{a/2}$ and $e^{(h+a)/2} = e^{h/2} e^{a/2}$ provided $[h, a] = 0$. Sum up,

$$e^{(h+a)/2} \not{D}^{g_0} e^{(h+a)/2} \backsim e^{h/2} \not{D}^{g_1} e^{h/2}.$$

From local expression side

- In g_1-calculus,

$$-K(\mathbf{x}_h)(\Delta^{g_1} h) + H(\mathbf{x}_h^{(1)}, \mathbf{x}_h^{(2)})(\nabla h \nabla h) \cdot g_1^{-1}$$

- In g_0-calculus,

$$-K(\mathbf{x}_h)(\Delta^{g_0}(h+a)) + H(\mathbf{x}_h^{(1)}, \mathbf{x}_h^{(2)})(\nabla(h+a)\nabla(h+a)) \cdot g_0^{-1}$$

- For them to be the same, crossed terms with $\nabla h \nabla a$ should cancel out with each other. They come from replacing $\Delta^{g_1} \to \Delta^{g_0}$ and expanding $\nabla(h+a)\nabla(h+a)$.
- Vanishing of the coefficients leads to the following equation (in $\dim M = 4$)

$$H(u,0) + H(0,u) = -2K(u)$$

- Unfortunately, this functional relation follows from those shown in a few slides before.