Einstein-Hilbert action on Connes-Landi noncommutative manifolds

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Motivations and History

Motivation:

- Explore notions of metric and curvature for certain class of noncommutative manifolds.
- Quantum gravity

On noncommutative two torus:

- Connes’s pseudo differential calculus 80’s. The computation was initiated at the same time;
- Gauss-Bonnet theorem (2011): Connes-Tretkoff, Khalkhali-Fathizadeh;

This talk is based on my work arXiv 1510.04668 (to appear in JNCG), arXiv 1611.08933.
Connes–Landi noncommutative manifolds $C^\infty(M_\theta)$

Let $M$ be a Riemannian manifold such that $\mathbb{T}^n \subset \text{Diff}(M)$.

$C^\infty(M)$ becomes a $\mathbb{T}^n$-module through the dual action: for $t \in \mathbb{T}^n$, $U_t : C^\infty(M) \rightarrow C^\infty(M)$:

$$U_t(f)(x) = f(t^{-1} \cdot x), \ \forall f \in C^\infty(M).$$

$\triangleright$ Isotypic decomposition: $\forall f \in C^\infty(M),$

$$f = \sum_{r \in \mathbb{Z}^n} f_r, \ \text{with} \ f_r = \int_{\mathbb{T}^n} e^{-2\pi i r \cdot t} U_t(f) dt.$$

In particular,

$$fh = \sum_{r,l} f_r h_l.$$
\(\theta\)-deformation (M. Rieffel)

- For any \(n \times n\) skew symmetric matrix \(\theta\), denote by bicharacter on \(\mathbb{Z} \times \mathbb{Z}\): \(\chi_\theta(r, l) = e^{2\pi i \langle \theta r, l \rangle}\).

- One can construct a new family of multiplication for \(C^\infty(M)\) by twisting the convolution: let \(f = \sum_r f_r\) and \(g = \sum_l g_l\),

\[
f \times_\theta h = \sum_{r,l} \chi_\theta(r, l)f_r h_l.
\]

- The new algebra \(C^\infty(M_\theta) \triangleq (C^\infty(M), \times_\theta)\) represents the “smooth structure” of \(M_\theta\).

- Natural vector bundles over \(M_\theta\) \((C^\infty(M_\theta)\)-bimodules\): \(f\) smooth function, \(X\) vector field, \(\omega\) one form: \(f \times_\theta X, \omega \times_\theta f, X \cdot_\theta \omega\) etc.

- Deforming linear operators \(P : C^\infty(M) \to C^\infty(M)\): \(\pi^\theta(P)(f) \triangleq P \times_\theta f\).
## Quantum mixed with Riemannian

### Quantum side

In general, the $\times_\theta$-multiplication is highly nonlocal. For example, the evaluation of $(f|_U) \times_\theta (h|_U)$ at some point $x \in U$ depends on $U$ in such a obscure way that the value has no compatibility when $U$ is shrinking around $x$.

### Riemannian side

If one of $h$ and $f$ is $\mathbb{T}^n$-invariant, then $f \times_\theta h = fh$. In particular,

- For a $\mathbb{T}^n$-invariant metric $g$, the associated connection and spinor Dirac are invariant under the deformation:
  \[
  \pi^\theta(\nabla) = \nabla, \quad \pi^\theta(\slashed{D}) = \slashed{D}.
  \]
- Leibniz property holds when computing $\nabla(X \cdot_\theta \omega)$.
- The spectral triple $(C^\infty(M_\theta), L^2(\$), \slashed{D})$ satisfies Connes’s axioms of spin geometry.
Scalar Curvature functional (Spectral Geometry)

For a given metric $g$ with the associated spinor Dirac $\slashed{D}$, the local geometry is encoded in the heat kernel asymptotic:

$$\text{Tr}(fe^{-t\slashed{D}^2}) \sim_{t \downarrow 0} \sum_{j=0}^{\infty} V_j(f, \slashed{D}^2) t^{\frac{j-m}{2}} \quad m = \text{dim } M.$$ 

- Coefficients are functionals: $f \in C^\infty(M) \rightarrow V_j(f, \slashed{D}^2 a) \in \mathbb{C}$.
- To “localize” the functionals, introduce $\varphi_0(\cdot) = \int_M \text{Tr}_x(\cdot)dg$
- Functional density $V_j(x) \in C^\infty(M)$ is defined by the property

$$V_j(f, \slashed{D}^2) = \varphi_0(f(x)V_j(x))$$

- In general, $V_j(x)$ is a polynomial whose arguments are components of the curvature tensor and the derivatives.
What we need

\[
V_2(x) = -\frac{1}{12} S_g.
\]

where \( S_g \) is the scalar curvature.

- \( V_2(\cdot, \mathcal{D}^2) \): scalar curvature functional.

The definition is intrinsic with respect to the spectral data:

- Define

\[
\int P = \text{Res}_{z=0} \text{Tr} \left( P \mathcal{D}^{-z} \right).
\]

- The functional \( \varphi_0 \) is proportional to

\[
\int (\cdot) \mathcal{D}^{-m}, \quad m = \text{dim } M.
\]

- The scalar curvature functional is proportional to

\[
\int (\cdot) \mathcal{D}^{-m+2}, \quad m = \text{dim } M.
\]
Symmetry Breaking

$\mathbb{T}^n$-invariant metrics behave like the Riemannian version

- The spectral triple $(C^\infty(M\theta), L^2(\$), \mathcal{D})$ satisfies Connes’s axioms of spin geometry.
- Leibniz property holds when computing $\nabla(X \cdot_\theta \omega)$.

The rest of the metrics seems intangible, because there is no straightforward calculus.

- Widom’s work tells us that any connection $\nabla$ will lead to a pseudo differential calculus.
- My work started with the observation: if $\nabla = \nabla^g$ for some $\mathbb{T}^n$-invariant metric $g$, such calculus can be deformed to study the spectral geometry of $M\theta$. 
Quantizing metrics

Let \( g' = e^{-2h}g \) for a real-valued \( h \in C^\infty(M) \). We would like to compare the two spinor Dirac \( \mathcal{D}^{g'} \) and \( \mathcal{D}^{g} \). There is a canonical and unitary identification between the \( L^2 \)-spinor sections

\[
\tilde{\beta}_{g^{-}g'} : L^2(g) \rightarrow L^2(g').
\]

**Proposition**

The \( g' \)-spinor Dirac \( \mathcal{D}^{g'} \) is unitary equivalent to

\[
\tilde{\beta}_{g^{-}g'} \mathcal{D}^{g'} \tilde{\beta}_{g} = e^{h/2} \mathcal{D}^{g} e^{h/2}.
\]
Conformal Geometry

Let us fix a $\mathbb{T}^n$-invariant metric $g$ as a background metric. The conformal class $[g]$ can be parametrized by $(g, h)$ with $h = h^* \in C^\infty(M)$. 

<table>
<thead>
<tr>
<th>$(M, g)$</th>
<th>$(C^\infty(M_\theta), L^2($), \mathcal{D})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-2h}$ with $h \in C^\infty(M_\theta)$ real-valued</td>
<td>$e^{-2h}$ with $h = h^* \in C^\infty(M_\theta)$ self-adjoint</td>
</tr>
<tr>
<td>$g' = e^{-2h}g$</td>
<td>$D_h = e^{h/2} \mathcal{D} e^{h/2}$</td>
</tr>
</tbody>
</table>

| Scalar curvature for $g'$: $S_{g'}$ | $R_{D_h} \in C^\infty(M_\theta)$: functional density of $V_2(\cdot, D_h^2)$ |

Table: Conformal change of metric and the associated scalar curvature in the noncommutative setting.
Modular scalar curvature

- Weyl factor (conformal factor), $k = e^h$ with $h = h^* \in C^\infty(M_\theta)$.
- Modular derivation (noncommutative differential):
  \[ x_h = -\text{ad}_h = [\cdot, h]. \]
  Modular operator: \[ y_h = \exp x_h = k^{-1}(\cdot)k. \]

**Theorem (Liu, 16)**

Let $\dim M = 4$. For $g' = e^{-2h}g$ or $D \mapsto D_h = e^{h/2}D e^{h/2}$, up to a constant factor, the modular scalar curvature defined in Table (1) is given by

\[
R_{g'} = e^{(-m+2)h} \left( K(x_h)(\nabla^2 h) + H(x_h^{(1)}, x_h^{(2)})(\nabla h \nabla h) \right) \cdot g^{-1} \\
+ cS_g
\]

Notations: write $H$ as a Fourier transform: $H = F \beta$,

\[
H(x_h^{(1)}, x_h^{(2)})(\psi \cdot \varphi) = \int_{\mathbb{R}^2} \beta(u, v)(y_h)^{-iu}(\psi)(y_h)^{-iv}(\varphi) du dv.
\]
The one-variable function $K$

Assume $\dim M = 4$.

\[ K(u) = -\frac{1}{2} e^u \frac{\sinh (u/2)}{(u/2)}, \]

which shows striking similarity to the AS local index formula.

- The $\hat{A}$-genus part comes from the fact that the deformation start on the tangent space of the metrics:

\[ \nabla e^h = f_1(ad_h)(\nabla h) = e^h \frac{e^{ad_h - 1}}{ad_h} (\nabla h). \]

- In local index formula, the $\hat{A}$-genus naturally through the Jacobian of the exponential map of a Lie group $G$: in exponential coordinates $d \exp h = j_g(h) dh$, where $\mathfrak{g}$ is the Lie algebra, $dh$ is a Haar measure and

\[ j_g = \det_g (-f_1(ad_h)) = \det_g \left( -\frac{e^{ad_h - 1}}{ad_h} \right). \]
Einstein Hilbert action

- Recall

\[ R_{g'} = e^{(-m+2)h} \left( K(x_h)(\nabla^2 h) + H(x^{(1)}_h, x^{(2)}_h)(\nabla h \nabla h) \right) \cdot g^{-1} + cS_g \]

- For each \( g' = e^{-2h}g \). We have the scalar curvature functional (on \( C^\infty(M) \)):

\[ f \mapsto V_2(f, D^2_h) = \varphi_0(fR_{g'}). \]

- The noncommutative analogue of the Einstein Hilbert action is given by

\[ F_{EH}(h) \triangleq V_2(1, D^2_h) \]

which has a local expression in dim four:

\[ F_{EH}(h) = \varphi_0(R_{g'}) = \varphi_0 \left( e^{-2h} [ T(x_h)(\nabla h) \cdot (\nabla h)] g^{-1} - \frac{1}{12} e^{-2h} S_g \right). \]
Local curvature functions in dim 4

- Modular scalar curvature:

\[
K(2u) = -\frac{2e^{-\frac{u}{2}} \sinh \left(\frac{u}{4}\right)}{u},
\]

\[
H(2u, 2v) = \frac{4e^{\frac{1}{4}(-3)(u+v)} \left( u \left(-e^{v/2}\right) + \left(e^{u/2} - 1\right) e^{v/2} v + u \right)}{uv(u + v)},
\]

\[c = K(0)/6.\]

- Einstein Hilbert action \(F_{EH}\):

\[
T(2u) = 2K(0)f_1(-2u) + H(2u, -2u)
\]

\[= \frac{e^{-u} - 1}{u} - \frac{-2u - 4e^{-\frac{u}{2}} + 4}{u^2}.\]

- Riemannian case is the commutative limit

\[K(0) = -\frac{1}{2}, H(0, 0) = \frac{1}{2}\]

\[-\frac{1}{12} e^{-m\hbar} S_{g'} = R_{g'} \bigg|_{x\hbar=0}\]
Variational Calculus

For any \( a = a^* \in C^\infty(M_\theta) \), for \( \varepsilon > 0 \), denote \( h_\varepsilon = h + \varepsilon a \). Set \( \delta_a \equiv d/d\varepsilon|_{\varepsilon=0} \). The gradient at \( h \): \( \text{grad}_h F_{EH} \) is define by the equation:

for any \( a = a^* \in C^\infty(M_\theta) \),

\[
\delta_a F_{EH}(h) = \varphi_0(a \text{grad}_h F_{EH}).
\]

By differentiating \( \delta_a \text{Tr}(e^{tD_h^2}) \), we obtain:

**Proposition**

*Let* \( m = \text{dim} \, M \),

\[
\delta_a V_2(1, D_\varepsilon^2) = \frac{2 - m}{2} V_2\left( \int_{-1}^{1} e^{uh/2}ae^{-uh/2}du, D_h^2 \right).
\]
Proposition

Let \( m = \dim M \) and \( j_0 = (m - 2)/2 \). The gradient \( \nabla_h F_{EH} \) is given by

\[
\nabla_h F_{EH} = e^{-2j_0 h} \left( K_{EH}(x_h)(\nabla^2 h) + H_{EH}(x_{h}^{(1)}, x_{h}^{(2)})(\nabla h \nabla h) \right) g^{-1} \\
+ (-2j_0)c(m)e^{-2j_0 h} S_\Delta,
\]

where

\[
K_{EH}(2s) = -8 j_0 \frac{\sinh(s/4)}{s} K(2s),
\]

\[
H_{EH}(2s, 2t) = -8 j_0 \frac{\sinh((s + t)/4)}{s + t} H(2s, 2t)
\]
On the other hand, we can compute the gradient $\text{grad}_h F_{EH}$ directly from the local expression

$$
\delta_a [\varphi_0(R_{g'})] = \delta_a \left[ \varphi_0 \left( e^{-2h} [T(x_h)(\nabla h) \cdot (\nabla h)] g^{-1} - \frac{1}{12} e^{-2h} S_g \right) \right].
$$

**Theorem (Liu, 16)**

The internal relations are given by

$$K_{EH}(s) = -(T + \tilde{T})(s),$$
$$H_{EH}(s, t) = (L + M - P - Q)(s, t),$$

where $\tilde{T}(s) = T(-s)e^{-s}$ and functions $L$, $M$, $P$ and $Q$ are given in the next page.
Some two-variable functions

- Recall: $f_1(u) = (e^u - 1)/u$ and $\tilde{T} = T(-s)e^{-s}$.
- Divided difference

\[ [u, v] T \triangleq \frac{T(u) - T(v)}{u - v}. \]

- Then

\[
\begin{align*}
L(s, t) &= -2T(s)f_1(-(s + t)), \\
M(s, t) &= 2e^{-(s+t)}(e^t[t, -s](T) - [s, -t](T)), \\
P(s, t) &= -2f_1(-s)T(t) - 2[s + t, t](T) + 2[s + t, s](T), \\
Q(s, t) &= -2f_1(-s)\tilde{T}(t) - 2[s + t, t](\tilde{T}) + 2[s + t, s](\tilde{T})
\end{align*}
\]

- (Lesch): the appearance of $-2[s + t, t](T) + 2[s + t, s](T)$ is due to the commutator of commutative and noncommutative derivations:

\[ [\nabla, T(x_h)](\psi). \]
Independence of the background metric

- Let $g'$ be the noncommutative metric in question, $g_0$ and $g_1$ are two background metrics ($\mathbb{T}^n$-invariant).
- $g' = e^{-2h}g_1 = e^{-2(h+a)}g_0$ with $a = a^* \in C^\infty(\mathcal{M}_\theta)$ commutative and $g_1 = e^{-2a}g_0$.
- We can quantize $g'$ in $g_1$ and $g_0$:
  
  $$ g' \mapsto e^{h/2} \mathcal{D}^{g_1} e^{h/2} \quad \text{or} \quad g' \mapsto e^{(h+a)/2} \mathcal{D}^{g_0} e^{(h+a)/2}. $$

- Since $\mathcal{D}^{g_1}$ is unitary equivalent to $e^{a/2} \mathcal{D}^{g_0} e^{a/2}$ and $e^{(h+a)/2} = e^{h/2} e^{a/2}$ provided $[h, a] = 0$. Sum up,

  $$ e^{(h+a)/2} \mathcal{D}^{g_0} e^{(h+a)/2} \sim e^{h/2} \mathcal{D}^{g_1} e^{h/2}. $$
From local expression side

- In $g_1$-calculus,
  
  $$-K(x_h)(\Delta^{g_1} h) + H(x_h^{(1)}, x_h^{(2)})(\nabla h \nabla h) \cdot g_1^{-1}$$

- In $g_0$-calculus,
  
  $$-K(x_h)(\Delta^{g_0} (h + a)) + H(x_h^{(1)}, x_h^{(2)})(\nabla (h + a) \nabla (h + a)) \cdot g_0^{-1}$$

- For them to be the same, crossed terms with $\nabla h \nabla a$ should cancel out with each other. They come from replacing $\Delta^{g_1} \rightarrow \Delta^{g_0}$ and expanding $\nabla (h + a) \nabla (h + a)$.

- Vanishing of the coefficients leads to the following equation (in $\dim M = 4$)
  
  $$H(u, 0) + H(0, u) = -2K(u)$$

- Unfortunately, this functional relation follows from those shown in a few slides before.