Sums of linear operators in Hilbert C*-modules

work in progress, based on discussions with Bram

Matthias Lesch

Universität Bonn
ml@uni-bonn.de

06.01.2017

Last Update: 2017-01.06

Outline

Hilbert C^*-modules

Unbounded operators

Regular operators

Localization

Local Global Principle

Sums of regular selfadjoint operators

Hilbert C^*–modules

Kaplansky 1953, Paschke 1973, Rieffel 1974, Kasparov 1980

1. $\mathcal{A}$ C^*-algebra
2. E Hilbert C^*–module over $\mathcal{A}$:
   - E $\mathcal{A}$–right module
   - $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathcal{A}$ inner product ($\mathcal{A}$–valued)
   - Banach space w.r.t. $\|x\| := \|\langle x, x\rangle\|^{1/2} = \|\langle x, x\rangle^{1/2}\|$
3. Superficially looks like a Hilbert space BUT
   - No Projection Theorem, hence closed submodules need not be complementable
   - No self-duality; unit ball is not (well, almost never) weakly compact.
4. $\mathcal{L}(E)$ bounded, adjointable, $\mathcal{A}$-module endomorphisms
   - $\mathcal{L}(E)$ is a C^*-algebra
   - Selfadjoint elements in $\mathcal{L}(E)$ do have a continuous functional calculus.
5. Why?
   - Important tool in Kasparov's bivariant KK-theory.
   - Curiosity driven: very natural generalization of Hilbert spaces

Examples of Hilbert C^*–modules ... are abundant

1. Hilbert space ($\mathcal{A} = \mathbb{C}$).
2. $E = \mathcal{A}$, $\langle x, y\rangle := x^* y$.
   $\mathcal{J} \subset \mathcal{A}$ closed non-trivial $*$–ideal. $\mathcal{J} \subset E$ closed non-trivial submodule.
   - E.g. $\mathcal{A} = C[0,1], \mathcal{J} = \{f \in \mathcal{A} \mid f(0) = 0\}$.
   - $\mathcal{J}^\perp = \{0\}$, $\mathcal{J}$ not complementable.
3. X compact space, $V \to X$ (continuous) vector bundle, h hermitian metric on V.
   - $\mathcal{A} := C(X)$; $E := \Gamma(X, V)$ (continuous sections of V)
   - $\langle f, g\rangle(x) := h(f(x), g(x))$
4. $H = \ell^2(\mathbb{N})$ standard Hilbert space,

$$H_{\mathcal{A}} = \left\{(a_j)_{j\in\mathbb{N}} \;\middle|\; \sum a_j^* a_j \text{ converges in } \mathcal{A}\right\}$$

$$\langle (a_j), (b_j)\rangle := \sum_{j=1}^{\infty} a_j^* b_j.$$

$H_{\mathcal{A}} = H \hat{\otimes}_{\mathcal{A}} \mathcal{A}$ standard module over $\mathcal{A}$.

Unbounded operators

At least: Banach space theory of unbounded operators available

T operator in E, domain $\mathcal{D}(T)$ dense in E

semiregular (operator affiliated with $\mathcal{A}$)

- $\mathcal{D}(T) \subset E$ dense submodule
- T^* densely defined

$\Rightarrow$ $T : \mathcal{D}(T) \to E$ $\mathcal{A}$–module map

Superficially looks like densely defined closable operator in Hilbert space.
Indeed: T closable, $T^* = \overline{T}^*$.

Pathologies

- No Functional Calculus for selfadjoint semiregular operators
- in general $\overline{T} \subsetneqq T^{**}$
- Exist $T = T^*$ semiregular, BUT $T + i\lambda$ not invertible.

Regular operators

Definition (Regular operator)

Let T semiregular. T regular if $I + T^*T$ has dense range.

Regular operators behave much like closable densely defined (resp. selfadjoint) operators in Hilbert space.

Proposition

Let T symmetric, densely defined, closed.

- T *regular* $\Leftrightarrow$ $T \pm i\,\mathrm{Id}$ *have complementable range.*
- T *selfadjoint and regular* $\Leftrightarrow$ $T \pm i\,\mathrm{Id}$ *have dense range.*

Proposition

Selfadjoint and regular operators admit a bounded continuous functional calculus, bounded imaginary powers etc. E.g. $\mathrm{spec}(T) \subset \mathbb{R}$, $f(T) \in \mathcal{L}(E), f \in C_b(\mathrm{spec}\, T), \|f(T)\| = \|f\|_\infty$, *in particular* $\|(T+z)^{-1}\| \leq 1/|\operatorname{Im} z|$,

Hilbert space case

$\mathcal{A} = \mathbb{C}$: semiregular $\Rightarrow$ regular

Localization

Exploit that for Hilbert spaces: "semiregular" $\Leftrightarrow$ "regular"

- $(\omega, H_\omega, \xi_\omega)$ cyclic representation of $\mathcal{A}$ w.r.t. state $\omega : \mathcal{A} \to \mathbb{C}$.
- $E \widehat{\otimes}_{\mathcal{A}} H_\omega$ Hilbert space completion of $E \otimes \xi_\omega \subset E \otimes_{\mathcal{A}} H_\omega$ w.r.t.

$$\langle x \otimes \xi_\omega, x' \otimes \xi_\omega \rangle = \omega\big(\langle x, x' \rangle_{\mathcal{A}}\big), \quad x, x' \in E$$

- $\mathcal{D}(T_0^\omega) := \mathcal{D}(T) \otimes_{\mathcal{A}} \xi_\omega \subset E \widehat{\otimes}_{\mathcal{A}} H_\omega$

$$T_0^\omega(x \otimes \xi_\omega) := (Tx) \otimes \xi_\omega \in E \widehat{\otimes}_{\mathcal{A}} H_\omega, \quad x \in \mathcal{D}(T).$$

Lemma

T_0^ω densely defined and closable. $(T^*)_0^\omega \subset (T_0^\omega)^*$.
$T^\omega := \overline{T_0^\omega}$ localization of T w.r.t. $(\omega, H_\omega, \xi_\omega)$.
Important: $(T^*)^\omega \subset (T^\omega)^*$.

E Hilbert $\mathcal{A}$–module.

Theorem A (Local–Global Principle; Pierrot 2006; Kaad-L 2012)

Let T closed semiregular operator in E.

$$T \text{ regular } \Leftrightarrow \quad \forall \text{ state } \omega \in S(\mathcal{A}): \ (T^*)^\omega = (T^\omega)^*.$$

Let additionally T be symmetric ($\langle Tx, y\rangle = \langle x, Ty\rangle$ for $x, y \in \mathcal{D}(T)$)

$$T \text{ selfadjoint and regular}$$
$$\Leftrightarrow \quad \forall \text{ state } \omega \in S(\mathcal{A}): \ \text{localization } T^\omega \text{ is selfadjoint.}$$

Theorem B (implicit in Pierot 2006; Kaad-L 2012)

Let $L \subset E, L \neq E$ closed nontrivial submodule; $x_0 \in E \setminus L$.

$$\exists \text{ state } \omega \in S(\mathcal{A}): \ x_0 \otimes \xi_\omega \notin \overline{L \otimes \xi_\omega}.$$

In particular $\exists$ state ω: $\big(L \otimes \xi_\omega\big)^\perp \neq \{0\}$.

Short: Submodule $L \subset E$ is dense $\Leftrightarrow \forall$ state ω: $L \otimes \xi_\omega$ dense in $E \widehat{\otimes} H_\omega$.

Application: weak cores

Proposition

T closed operator in Hilbert space $\mathcal{H}$

(A) *Let $(x_n) \subset \mathcal{D}(T)$ with $x_n \rightharpoonup x$ (weak convergence), $\sup_n \|Tx_n\| < \infty$. Then $x \in \mathcal{D}(T)$ and $Tx_n \rightharpoonup Tx$.*

(B) *Let $\mathcal{E} \subset \mathcal{D}(T)$ subspace. Let $\widetilde{\mathcal{E}}$ space of $x \in \mathcal{D}(T)$ admitting an approximating sequence $(x_n) \subset \mathcal{E}$ as in (A). If $\widetilde{\mathcal{E}}$ is a core for T then $\mathcal{E}$ is a core for T.*

Proposition

Let T be a semi-regular operator in the Hilbert $\mathcal{A}$-module E.
In general (A) fails, even if T is regular.
(B) holds true if $\mathcal{E}$ is a submodule

Proof 1

1. For $y \in \mathcal{D}(T^*)$:

$$|\langle x, Ty\rangle| = \lim_n |\langle x_n, Ty\rangle| = \lim_n |\langle Tx_n, y\rangle| \le \left(\sup_n \|Tx_n\|\right) \cdot \|y\|,$$

thus $x \in \mathcal{D}(T)$ and

$$\langle Tx, y\rangle = \langle x, Ty\rangle = \lim_n \langle x_n, Ty\rangle = \lim_n \langle Tx_n, y\rangle, y \in \mathcal{D}(T),$$

thus $Tx_n \rightharpoonup Tx$.

2. $\Gamma(\mathcal{E})$:=graph of T over $\mathcal{E}$, similarly $\Gamma(\widetilde{\mathcal{E}})$.

$$\overline{\Gamma(\mathcal{E})}^{\text{strong}} \subset \Gamma(\widetilde{\mathcal{E}}) \subset \overline{\Gamma(\mathcal{E})}^{\text{weak}} = \overline{\Gamma(\mathcal{E})}^{\text{strong}}$$

$\widetilde{\mathcal{E}}$ core $\Rightarrow \overline{\Gamma(\mathcal{E})}^{\text{strong}} = \overline{\Gamma(\widetilde{\mathcal{E}})} = \Gamma(T)$.

Proof 2

3. Counterexample to (A) for Hilbert modules: $\mathcal{A} = C_b(\mathbb{N}), E = C_0(\mathbb{N})$, $\langle f, g\rangle(k) := \overline{f(k)} \cdot g(k)$.

- $Tf(k) := k \cdot f(k)$ if $(k \cdot f(k))_k$ is bounded.
- T is selfadjoint and regular.
- Fix $F \in C_b(\mathbb{N})$ such that $\lim_{k\to\infty} F(k)$ does not exist.

$$f_n(k) := \begin{cases} \frac{1}{k} \cdot F(k) & k \leq n, \\ 0 & k > n. \end{cases}$$

- Then $f_n \in \mathcal{D}(T)$, $f_n \to f := \frac{1}{\mathrm{id}} F \in E$, $\|Tf_n\| \leq \|F\|_\infty$. BUT $f \notin \mathcal{D}(T)$.
- **Nevertheless**: $C_c(\mathbb{N})$ is a core for T.

Proof 3

4. Proof of (B) using Theorem (B):
Fix state ω with cyclic representation (H_ω, ξ_ω).

- $\mathcal{D}(T^\omega) = \mathcal{D}(T)\widehat{\otimes} H_\omega$.
- Fix $x \otimes \xi_\omega \in \mathcal{D}(T) \otimes \xi_\omega$, choose sequence $(x_n) \subset \mathcal{E}$ with $x_n \rightharpoonup x$ and $\sup_n \|Tx_n\| < \infty$. For any $\eta \in E\widehat{\otimes} H_\omega$

$$E \ni z \mapsto \langle \eta, z \otimes \xi_\omega \rangle_{E\widehat{\otimes} H_\omega}$$

  is continuous linear form, hence

$$\langle x \otimes \xi_\omega, \eta \rangle_{E\widehat{\otimes} H_\omega} = \lim_n \langle x_n \otimes \xi_\omega, \eta \rangle_{E\widehat{\otimes} H_\omega},$$

  thus $x_n \otimes \xi_\omega \rightharpoonup x \otimes \xi_\omega$ in $E\widehat{\otimes}\mathcal{H}_\omega$.
- $\sup_n \|T^\omega(x_n \otimes \xi_\omega)\| = \sup_n \|\omega(\langle Tx_n, Tx_n\rangle)\| \leq \sup_n \|Tx_n\|^2 < \infty$.
- Result: $\forall \omega : \mathcal{E} \otimes \xi_\omega$ is dense in $\mathcal{D}(T^\omega) = \mathcal{D}(T)\widehat{\otimes} H_\omega$.
  With Theorem B: $\mathcal{E}$ is dense in $\mathcal{D}(T)$.

Sums of regular selfadjoint operators

Motivation Unbounded KK-product

- $D_1 \hat{\otimes} D_2 = D_1 \otimes 1 + 1 \otimes_\nabla D_2$
- Example: $(A(t))_{t\in\mathbb{R}}$ family of selfadjoint Fredholm operators in H_1 (single operator in $H = H_1 \hat{\otimes}_{C_0(\mathbb{R})} C_0(\mathbb{R})$). In $H \hat{\otimes}_{C_0(\mathbb{R})} L^2(\mathbb{R}) = H_1 \hat{\otimes} L^2(\mathbb{R})$:

$$\underbrace{\begin{pmatrix} 0 & A(t) \\ A(t) & 0 \end{pmatrix}}_{S} + \underbrace{\begin{pmatrix} 0 & \frac{d}{dx} \\ -\frac{d}{dx} & 0 \end{pmatrix}}_{T}$$

S, T selfadjoint regular operators in E

Problem

Appropriate smallness condition on $[S, T] = ST + TS$ should imply $S + T$ selfadjoint and regular

- Both operators are sectorial with spectral angle π (hyperbolic case)
- S^2, T^2 are nonnegative (sectorial) operators

Banach space results

Theorem (Da Prato-Grisvard)

A, B sectorial operators in a Banach space X with spectral angle $< \pi$.
Assume $(A+\lambda)^{-1}\mathcal{D}(B) \subset \mathcal{D}(B)$ and

$$\begin{aligned}\big\| B(A+\lambda)^{-1} - (A+\lambda)^{-1}B\big)(\mu+B)^{-1}\big\| \\ \leq \frac{c}{(1+|\lambda|)^{\alpha}|\mu|^{\beta}}, \quad \alpha,\beta>0, \beta<1, \alpha+\beta>1\end{aligned}$$

Then for λ large enough (outside spectral sector) $\overline{A+B}+\lambda$ is invertible.

- Labbas-Terreni: same conclusion under

$$\begin{aligned}\|A(A+\lambda)^{-1}\big(A^{-1}(B+\mu)^{-1} - (B+\mu)^{-1}A^{-1}\big)\| \\ \leq \frac{c}{(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}}; \quad 0 \leq \alpha < \beta < 1\end{aligned}$$

Banach space results II

- Closedness of $A + B$ on $\mathcal{D}(A) \cap \mathcal{D}(B)$ proved under additional assumptions on the Banach space X ($\mathcal{HT}$) and that A, B admit $\mathcal{BIP}$.

  Dore-Venni 1987 A, B, resolvent commuting

  Monnieux-Prüss 1997 A, B satisfy Labbas-Terreni commutator condition

  Prüss-Simonett 2007 Da Prato-Grisvard or Labbas-Terreni commutator condition; emphasis on H^∞ calculus.

- Important pattern:

$$P_\lambda := \frac{1}{2\pi i} \int_\Gamma (z + \lambda + A)^{-1} \cdot (z - B)^{-1} dz$$

"approximates" $(A + B + \lambda)^{-1}$.

Main Result

Theorem C

S, T selfadjoint and regular in Hilbert $\mathcal{A}$–module E.
Assumptions:

1. $(S+\lambda)^{-1}\big(\mathcal{D}(T)\big) \subset \mathcal{F}$ for $\lambda \in i\mathbb{R}$ large enough.

$$\mathcal{F} := \mathcal{F}(S,T) := \big\{x \in \mathcal{D}(S) \cap \mathcal{D}(T) \;\big|\; Sx \in \mathcal{D}(T), Tx \in \mathcal{D}(S)\big\}$$

2. For $x \in \mathcal{F}$:

$$\big\|[S,T]x := (S \cdot T + T \cdot S)x\| \leq C_1 \cdot \|Sx\| + C_2 \cdot \|Tx\| + C_3 \cdot \|x\|.$$

Then $S + T$ with domain $\mathcal{D}(S) \cap \mathcal{D}(T)$ is selfadjoint and regular.

In equation solver speak (more impressive): for $z \in \mathbb{C} \setminus \mathbb{R}$ and $y \in E$ the equation

$$Sx + Tx + z \cdot x = y$$

has a unique solution $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$.

Main Result

Theorem C

S, T selfadjoint and regular in Hilbert $\mathcal{A}$–module E.
Assumptions:

1. $(S+\lambda)^{-1}\big(\mathcal{D}(T)\big) \subset \mathcal{F}$ for $\lambda \in i\mathbb{R}$ large enough.

$$\mathcal{F} := \mathcal{F}(S,T) := \big\{x \in \mathcal{D}(S) \cap \mathcal{D}(T) \;\big|\; Sx \in \mathcal{D}(T),\, Tx \in \mathcal{D}(S)\big\}$$

2. For $x \in \mathcal{F}$:

$$\big\|[S,T]x := (S \cdot T + T \cdot S)x\| \leq C_1 \cdot \|Sx\| + C_2 \cdot \|Tx\| + C_3 \cdot \|x\|.$$

Then $S + T$ with domain $\mathcal{D}(S) \cap \mathcal{D}(T)$ is selfadjoint and regular.

Remark

1. *The assumptions are symmetric in S, T (exchange roles of S, T).*
2. *Suffices that $(S+\lambda)^{-1}(\mathcal{E}) \subset \mathcal{F}$ and (2) holds on $(S+\lambda)^{-1}(\mathcal{E})$ for a core $\mathcal{E}$ of T.*

Comparison to da Prato Grisvard / Labbas-Terreni

1. $\|[S, T](S+\lambda)^{-1}(T+\mu)^{-1}\| \leq c\left(\frac{1}{|\lambda|}+\frac{1}{|\mu|}\right)$.
2. $\Rightarrow$

$$[T^2,(S^2+\lambda)^{-1}](T^2+\mu)^{-1} \ll \frac{1}{|\lambda|}\left(\frac{1}{\sqrt{|\lambda|}}+\frac{1}{\sqrt{|\mu|}}\right),$$

BUT $(S^2+\lambda)^{-1}\mathcal{D}(T^2) \not\subset \mathcal{D}(T^2)$.

3.

$$P_\lambda := \frac{1}{2\pi i}\int_\Gamma (z+\lambda+S^2)^{-1}(S+T-i\lambda)(z-T^2)^{-1}dz$$

For $y \in \mathcal{D}(S)\cap\mathcal{D}(T), \lambda$ large

$$(S+T+i\lambda)P_\lambda y = (I+R_\lambda)y, \|R_\lambda\| < 1,$$

hence $ran(S+T+i\lambda)$ dense.

Main Result: Consequences

1. $(S+\lambda)^{-1}\mathcal{D}(T) \subset \mathcal{F}$ and $(T+\lambda)^{-1}\mathcal{D}(S) \subset \mathcal{F}$ for all $\lambda \in i\mathbb{R}, |\lambda| \geq \lambda_0$.
2. For $\lambda, \mu \in i\mathbb{R}, |\lambda, \mu| \geq \lambda_0$

$$\operatorname{ran}(T+\mu)^{-1} \cdot (S+\lambda)^{-1} = \operatorname{ran}(S+\lambda)^{-1} \cdot (T+\mu) = \mathcal{F}$$

3. $\mathcal{D}(S) \cap \mathcal{D}(T)$ is dense in E and $\mathcal{F}$ is dense in $\mathcal{D}(S) \cap \mathcal{D}(T)$ in the following sense: for $x \in \mathcal{D}(S) \cap \mathcal{D}(T)$,

$$x_\lambda := \lambda^2 (T+\lambda)^{-1} \cdot (S+\lambda)^{-1} x \in \mathcal{F},$$

and $x_\lambda \to x, Sx_\lambda \to Sx$, $Tx_\lambda \to Tx$, as $i\mathbb{R} \ni \lambda \to \infty$.

Theorem
$S^2 + T^2$ is selfadjoint and regular on $\mathcal{D}(S^2) \cap \mathcal{D}(T^2) = \mathcal{D}((S+T)^2)$. For $x \in \mathcal{D}(S^2) \cap \mathcal{D}(T^2)$ one has automatically $Sx \in \mathcal{D}(T)$, $Tx \in \mathcal{D}(S)$.

Application: Iteration

Theorem

S_1, S_2, S_3 selfadjoint and regular. Assume that $(S_1, S_2), (S_2, S_3), (S_1, S_3)$ satisfy the assumptions of Theorem C. Then also $(S_1 + S_2, S_3)$ satisfies these assumptions and $S_1 + S_2 + S_3$ is selfadjoint and regular on $\mathcal{D}(S_1) \cap \mathcal{D}(S_2) \cap \mathcal{D}(S_3)$.

Hard

$$(S_1 + S_2 + \lambda)^{-1}\mathcal{D}(S_3) \subset \mathcal{F}(S_1 + S_2, S_3).$$

Easy

$$(S_3 + \lambda)^{-1}\big(\mathcal{D}(S_1) \cap \mathcal{D}(S_2)\big) \subset \mathcal{F}(S_1 + S_2, S_3)$$

Structure of Proof

1. Extend domain inclusion to all $|\lambda, \mu| \geq |\lambda_0|$

$$(S+\lambda)(T+\mu) - (T+\mu)(S+\lambda) = ST - TS$$

   Use Clifford Algebra trick.

2. Closedness of Sum operator:

$$c_1\big(\|Sx\| + \|Tx\| + \|x\|\big) \leq \|(S+T)x\| + \|x\| \leq \|Sx\| + \|Tx\| + \|x\|$$

3. Selfadjointness: $\lambda^2(S+\lambda)^{-1}(T+\lambda)^{-1}$
4. Regularity: Local Global Principle

Clifford Algebra tool

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \omega := i\sigma_1 \cdot \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

generators of $\mathbb{C}\ell(2)$. Replace E by $E \otimes \mathbb{C}^2$ (ungraded), S, T by

$$S \otimes I = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \quad T \otimes I = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}.$$

$\Rightarrow$**W.l.o.g.** $\mathbb{C}\ell(2) \subset \mathcal{L}(E)$ acts unitarily on E and commutes with S, T.
Relations:

$$(S\sigma_j)^* = S\sigma_j, \quad (T\sigma_j)^* = T\sigma_j, \tag{6.1}$$

$$S\sigma_1 \cdot T\sigma_2 - T\sigma_2 \cdot S\sigma_1 = (ST + TS)\sigma_1\sigma_2 \tag{6.2}$$

$$S\omega \cdot T + T \cdot S\omega = (ST + TS) \cdot \omega \tag{6.3}$$

$$(S \cdot \omega + T) \cdot S\sigma_1 + S\sigma_1 \cdot (S \cdot \omega + T) = (ST + TS) \cdot \sigma_1 \tag{6.4}$$

Closedness of the sum operator

$$\langle (S+T)x,(S+T)x\rangle = \langle Sx,Sx\rangle + \langle Tx,Tx\rangle + \underbrace{\langle Sx,Tx\rangle + \langle Tx,Sx\rangle}_{\leq \langle Sx,Sx\rangle + \langle Tx,Tx\rangle}$$

$$\begin{aligned}\langle Sx,Sx\rangle + \langle Tx,Tx\rangle &= \frac{1}{2}\big(\langle \frac{1}{\mu}[S,T]x,\mu x\rangle + \langle \mu x,\frac{1}{\mu}[S,T]x\rangle\big)\\ &\leq \mu^{-2}C_1\|Sx\|^2 + \mu^{-2}C_2\|Tx\|^2 + \mu^2C_3\|x\|^2\\ &\leq \frac{1}{4}\|\langle Sx,Sx\rangle\| + \frac{1}{4}\|\langle Tx,Tx\rangle\| + C\|x\|^2\\ &\leq \frac{1}{2}\|\langle Sx,Sx\rangle + \langle Tx,Tx\rangle\| + C\|x\|^2\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\|\langle Sx,Sx\rangle\| + \|\langle Tx,Tx\rangle\| &\leq 2\|\langle Sx,Sx\rangle + \langle Tx,Tx\rangle\|\\ &\leq 4\|\langle (S+T)x,(S+T)x\rangle\| + C\|x\|^2\\ &\leq 8\|\langle Sx,Sx\rangle\| + 8\|\langle Tx,Tx\rangle\| + C\|x\|^2.\end{aligned}$$

Proof of Selfadjointness and Regularity

Selfadjointness $x \in \mathcal{D}((S+T)^*)$;

$$\begin{aligned} x_\lambda &:= \lambda^2 (S+\lambda)^{-1}(T+\lambda)^{-1}x \in \mathcal{F} \\ x_\lambda &\to x \\ (S+T)x_\lambda &= \text{Commutator Term} \\ &\qquad + \lambda^2 (S+\lambda)^{-1}(T+\lambda)^{-1}(S+T)^* x \to (S+T)^* x. \end{aligned}$$

Regularity All Localizations $S^\omega + T^\omega = (S+T)^\omega$ selfadjoint **Local Global Principle** $\Rightarrow$ $S+T$ regular.