On a theorem of Kucerovsky for half-closed chains

Jens Kaad

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Polar decomposition

Definition

1. *Vertical data:* $(E, \frac{i}{r}\frac{\partial}{\partial\theta}) : C_c^\infty(\mathbb{R}^2 \setminus \{0\}) \to C_0\big((0,\infty)\big)$.
   Unbounded Kasparov module.

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2. *Horizontal data:* $\big(L^2\big((0,\infty)\big), i\frac{d}{dr}\big) : C_c^\infty\big((0,\infty)\big) \to \mathbb{C}$. *Half closed chain.*

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Question

Is it true that

$$\iota^*[\mathbb{R}^2] = [E, \frac{i}{r}\frac{\partial}{\partial\theta}] \widehat{\otimes}_{C_0\left((0,\infty)\right)} [L^2\big((0,\infty)\big), i\frac{d}{dr}]$$

in $K^0\big(C_0(\mathbb{R}^2 \setminus \{0\})\big)$*?*

KK-theory

Notation

1. *A, B, C, etc. are separable C^*-algebras.*
2. *Kasparov's bivariant K-theory denoted by $KK(A, B)$.*
3. *Classes in KK-theory are given by Kasparov modules $(E, F) : A \to B$ up to homotopy.*

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Theorem (Kasparov)

There exists a bilinear and associative pairing

$$\widehat{\otimes}_B : KK(A,B) \times KK(B,C) \to KK(A,C)$$

How to recognize a Kasparov product?

Theorem (Connes, Skandalis)

Suppose that $(E = E_1 \widehat{\otimes}_B E_2, F) : A \to C$, $(E_1, F_1) : A \to B$ *and* $(E_2, F_2) : B \to C$ *are Kasparov modules. Suppose moreover that*

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1. $$T_\xi F_2 - (-1)^{\partial \xi} F T_\xi : E_2 \to E_1 \widehat{\otimes}_B E_2$$
   is compact for all homogeneous $\xi \in E_1$.
2. $$a^* \cdot \big(F(F_1 \widehat{\otimes} 1) + (F_1 \widehat{\otimes} 1) F \big) \cdot a : E_1 \widehat{\otimes}_B E_2 \to E_1 \widehat{\otimes}_B E_2$$
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   is positive modulo compacts for all $a \in A$.

Then $[E_1 \widehat{\otimes}_B E_2, F] = [E_1, F_1] \widehat{\otimes}_B [E_2, F_2]$ *in* $KK(A, C)$.

Half closed chains (I)

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$\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, etc. are $*$-algebras equipped with fixed C^*-norms and C^*-completions A, B, C, etc.

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1. *A countably generated C^*-correspondence E from A to B;*
2. *A symmetric and regular unbounded operator $D : \mathrm{Dom}(D) \to E$, such that:*

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Definition (Continued)

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Remark

The **unbounded Kasparov modules** *from* $\mathscr{A}$ *to* B *are exactly the* **half closed chains** (E, D) *from* $\mathscr{A}$ *to* B *with* $D = D^*$.

Elliptic differential operators

Theorem (Baum, Douglas, Taylor)

Let M be **any** *Riemannian manifold and let $D : \Gamma_c^\infty(M, E) \to L^2(M, E)$ be* **any** *symmetric and elliptic first order differential operator with closure $\overline{D}$. Then*

$$(L^2(M, E), \overline{D}) : C_c^\infty(M) \to \mathbb{C}$$

is a half closed chain. Moreover, when M is **complete** *and D has* **finite propagation speed** *this half closed chain is a spectral triple.*

Bounded transforms of half closed chains

Theorem (Hilsum)

Suppose that $(E, D) : \mathscr{A} \to B$ *is a half closed chain from* $\mathscr{A}$ *to* B*. Then* $(E, F_D) := (E, D(1 + D^* D)^{-1/2}) : A \to B$ *is a Kasparov module.*

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Definition

The **Baaj-Julg-bounded transform** *of a half closed chain* $(E, D) : \mathscr{A} \to B$ *is the class* $[E, F_D] \in KK(A, B)$*.*

Kucerovsky's theorem for unbounded Kasparov modules

Theorem

Let $(E_1 \widehat{\otimes}_B E_2, D) : \mathscr{A} \to C$, $(E_1, D_1) : \mathscr{A} \to B$ *and* $(E_2, D_2) : \mathscr{B} \to C$ *be* **unbounded Kasparov modules**. *Suppose that*

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1. *There exists a dense* $\mathscr{B}$*-submodule* $\mathscr{E}_1 \subseteq E$ *such that*

$$DT_\xi - (-1)^{\partial \xi} T_\xi D_2 : \mathrm{Dom}(D_2) \to E_1 \widehat{\otimes}_B E_2$$

   is bounded adjointable for all homogeneous $\xi \in \mathscr{E}_1$.

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2. $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D_1 \widehat{\otimes} 1)$ *and there exists* $\kappa \in (0, \infty)$ *such that*

$$\langle D\xi, (D_1 \widehat{\otimes} 1)\xi \rangle + \langle (D_1 \widehat{\otimes} 1)\xi, D\xi \rangle \geq -\kappa \langle \xi, \xi \rangle$$

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Then $[E_1 \widehat{\otimes}_B E_2, F_D] = [E_1, F_{D_1}] \widehat{\otimes}_B [E_2, F_{D_2}]$ *in* $KK(A, C)$.

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3. *A bounded positive operator* $\Delta : E \to E$ *with dense image, such that:*

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$$D(a+\lambda)\Delta(\xi) - \Delta(a+\lambda)D(\xi) = \Delta^{1/2} d_\Delta(a,\lambda)\Delta^{1/2}(\xi)$$

for some bounded operator $d_\Delta(a,\lambda) : E \to E$*.*

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3. *The sequence* $\{a \cdot \Delta(\Delta + 1/n)^{-1}\}$ *converges in operator norm to* a *for all* $a \in A$*.*

Bounded transforms of unbounded modular cycles

Theorem (K.)

Suppose that $(E, D, \Delta) : \mathscr{A} \to B$ *is an unbounded modular cycle. Then* $(E, F_D) := (E, D(1 + D^2)^{-1/2}) : A \to B$ *is a Kasparov module.*

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Theorem (K., Suijlekom)

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Then the **localization** $(\overline{xE}, D_x, xx^*) : x\mathscr{A}x^* \to B$ *is an unbounded modular cycle. In particular, D_x is a* **selfadjoint** *and regular unbounded operator.*

Localization as a Kasparov product

Theorem (K., Suijlekom)

Let $(E, D) : \mathscr{A} \to B$ be a half closed chain and let $x \in \mathscr{A}$. Then

$$[E_x, F_{D_x}] = [\overline{xA}, 0]\widehat{\otimes}_A[E, F_D]$$

in $KK(\overline{xAx^*}, B)$.

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Let $(E_1 \widehat{\otimes}_B E_2, D) : \mathscr{A} \to C$, $(E_1, D_1) : \mathscr{A} \to B$ *and* $(E_2, D_2) : \mathscr{B} \to C$ *be* **half closed chains***. Suppose exist* **dense** $\mathscr{B}$**-submodule** $\mathscr{E}_1 \subseteq E_1$ *and a* **countable generating subset** $\Lambda \subseteq \mathscr{A}_{\mathrm{sa}}$*:*

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