Riemannian principal bundles in unbounded *KK*-theory

Branimir Ćaćić [1] and Bram Mesland [2,3]

Analysis, Noncommutative Geometry, and Operator Algebras
Chalmers University of Technology

June 12, 2017

[1]Department of Mathematics & Statistics, University of New Brunswick

[2]Institute for Analysis, Leibniz Universität Hannover

[3]Mathematical Institute, University of Bonn

Back to basics

The setup

Suppose that:

- P is a compact oriented n-manifold,
- G is a compact connected Lie group of dimension m,
- $G \to \mathrm{Diff}^+(P)$ is a free orientation-preserving Lie action.

The setup

Suppose that:

- P is a compact oriented n-manifold,
- G is a compact connected Lie group of dimension m,
- $G \to \mathrm{Diff}^+(P)$ is a free orientation-preserving Lie action.

Then:

- $B := P/G$ is a compact orientable $(n-m)$-manifold,
- $\pi : P \twoheadrightarrow B$ defines a principal G-bundle.

The setup

Suppose that:

- P is a compact oriented n-manifold,
- G is a compact connected Lie group of dimension m,
- $G \to \mathrm{Diff}^+(P)$ is a free orientation-preserving Lie action.

Then:

- $B := P/G$ is a compact orientable $(n-m)$-manifold,
- $\pi : P \twoheadrightarrow B$ defines a principal G-bundle.

Question

What if P is given a G-invariant Riemannian metric g^{TP}?

The horizontal geometry

The HORIZONTAL TANGENT BUNDLE

$$HP := \sqcup_{p \in P} T_p \left(G \cdot p \right)^{\perp} = (\ker d\pi)^{\perp}$$

is a trivially G-equivariant Riemannian vector bundle.

The horizontal geometry

The HORIZONTAL TANGENT BUNDLE

$$HP := \sqcup_{p \in P} T_p \left(G \cdot p\right)^{\perp} = (\ker d\pi)^{\perp}$$

is a trivially G-equivariant Riemannian vector bundle.

1. The derivative $d\pi : TP \to TB$ restricts to $HP \twoheadrightarrow HP/G \cong TB$.
2. The metric $g^{TP}|_{HP}$ pushes forward to a metric g^{TB} on B, making $\pi : P \twoheadrightarrow B$ a Riemannian submersion.
3. There exists a unique G-invariant metric connection ∇^{HP} on HP pushing forward to the Levi-Civita connection on B.

The horizontal geometry

The HORIZONTAL TANGENT BUNDLE

$$HP := \sqcup_{p \in P} T_p \, (G \cdot p)^{\perp} = (\ker d\pi)^{\perp}$$

is a trivially G-equivariant Riemannian vector bundle.

1. The derivative $d\pi : TP \to TB$ restricts to $HP \twoheadrightarrow HP/G \cong TB$.
2. The metric $g^{TP}|_{HP}$ pushes forward to a metric g^{TB} on B, making $\pi : P \twoheadrightarrow B$ a Riemannian submersion.
3. There exists a unique G-invariant metric connection ∇^{HP} on HP pushing forward to the Levi-Civita connection on B.

Lesson

You can isolate the HORIZONTAL GEOMETRY of $P \twoheadrightarrow B$ *upstairs* and then push it forward to get the Riemannian geometry of B.

The vertical geometry

The VERTICAL TANGENT BUNDLE

$$VP := \sqcup_{p \in P} T_p \left(G \cdot p \right) = \ker d\pi$$

is a non-trivially G-equivariant Riemannian vector bundle.

The vertical geometry

The VERTICAL TANGENT BUNDLE

$$VP := \sqcup_{p\in P} T_p\,(G\cdot p) = \ker d\pi$$

is a non-trivially G-equivariant Riemannian vector bundle.

- Differentiating $G \curvearrowright P$ yields G-equivariant $\mathfrak{g} \times P \xrightarrow{\sim} VP$.
- The orthogonal projection $\Pi^{VP} : TP \twoheadrightarrow VP$ defines a principal connection on $P \twoheadrightarrow B$.
- Compressing the Levi-Civita connection on TP yields a G-equivariant metric connection ∇^{VP} on VP.

The vertical geometry

The VERTICAL TANGENT BUNDLE

$$VP := \sqcup_{p \in P} T_p\left(G \cdot p\right) = \ker d\pi$$

is a non-trivially G-equivariant Riemannian vector bundle.

- Differentiating $G \curvearrowright P$ yields G-equivariant $\mathfrak{g} \times P \xrightarrow{\sim} VP$.
- The orthogonal projection $\Pi^{VP} : TP \twoheadrightarrow VP$ defines a principal connection on $P \twoheadrightarrow B$.
- Compressing the Levi-Civita connection on TP yields a G-equivariant metric connection ∇^{VP} on VP.

Lesson

The VERTICAL GEOMETRY of $P \twoheadrightarrow B$ encodes the principal G-action on P and the Chern–Weil theory of $P \twoheadrightarrow B$.

“Factorising” the total geometry

In terms of G-equivariant Riemannian vector bundles,

$$TP = VP \oplus HP \cong (\mathfrak{g} \times P) \oplus \pi^* TB,$$

“Factorising” the total geometry

In terms of G-equivariant Riemannian vector bundles,

$$TP = VP \oplus HP \cong (\mathfrak{g} \times P) \oplus \pi^* TB,$$

but in terms of G-equivariant metric connections,

$$\nabla^{TP} - S^{TP} = \nabla^{VP} \oplus \nabla^{HP},$$

"Factorising" the total geometry

In terms of G-equivariant Riemannian vector bundles,

$$TP = VP \oplus HP \cong (\mathfrak{g} \times P) \oplus \pi^* TB,$$

but in terms of G-equivariant metric connections,

$$\nabla^{TP} - S^{TP} = \nabla^{VP} \oplus \nabla^{HP},$$

where the G-invariant $\mathrm{End}(TP)$-valued 1-form S^{TP} encodes:

- the curvature 2-form $\Omega \in \Gamma(\wedge^2 HP^* \otimes VP)^G$ of Π^{VP},
- the mean curvature 1-form $\kappa \in \Omega^1(P)$ of $\pi : P \twoheadrightarrow B$.

$$\begin{pmatrix}\text{total}\\ \text{geometry}\end{pmatrix} - \begin{pmatrix}\text{curvature}\\ \text{information}\end{pmatrix} = \begin{pmatrix}\text{vertical}\\ \text{geometry}\end{pmatrix} \oplus \begin{pmatrix}\text{horizontal}\\ \text{geometry}\end{pmatrix}$$

$$\pi_* \begin{pmatrix}\text{horizontal}\\ \text{geometry}\end{pmatrix} = \begin{pmatrix}\text{geometry}\\ \text{of the base}\end{pmatrix}$$

$$\begin{pmatrix}\text{vertical}\\ \text{geometry}\end{pmatrix} \supset \begin{pmatrix}\text{principal}\\ \text{Lie action}\end{pmatrix} \cup \begin{pmatrix}\text{Chern–Weil}\\ \text{theory}\end{pmatrix}$$

Tricks with Dirac bundles

Relevant literature

Relevant literature

Transversal Dirac operators for Riemannian foliations

- J. F. Glazebrook and F. W. Kamber (1991)

Relevant literature

Transversal Dirac operators for Riemannian foliations

- J. F. Glazebrook and F. W. Kamber (1991)

Families index theory of Riemannian submersions

- J.-M. Bismut (1986)
- N. Berline and M. Vergne (1987)
- J. Kaad and W. D. van Suijlekom (2016)

Relevant literature

Transversal Dirac operators for Riemannian foliations

- J. F. Glazebrook and F. W. Kamber (1991)

Families index theory of Riemannian submersions

- J.-M. Bismut (1986)
- N. Berline and M. Vergne (1987)
- J. Kaad and W. D. van Suijlekom (2016)

Dirac operators and Riemannian principal bundles

- B. Ammann and C. Bär (1998)
- A. Moroianu (1996)
- S. Hong (2016, 2017)

Dirac bundles

Definition

A DIRAC BUNDLE on (M, g^{TM}) is a triple (E, c^E, ∇^E), where:

- $E \to M$ is a $\mathbf{Z}/2\mathbf{Z}$-graded Hermitian vector bundle;
- $c^E : \mathbf{C}\mathrm{l}(T^*M) \widehat{\otimes} \mathbf{C}\mathrm{l}_{\dim M} \to \mathrm{End}(E)$ is an even fibrewise $*$-monomorphism;
- ∇^E is an even Hermitian connection on E such that

$$\forall X \in \mathfrak{X}(M),\ \forall \omega \in \Omega^1(M), \quad [\nabla^E_X, c^E(\omega)] = c^E(\nabla^{T^*M}_X \omega), \tag{1}$$

$$\forall X \in \mathfrak{X}(M),\ \forall y \in \mathbf{C}\mathrm{l}_n, \quad [\nabla^E_X, c^E(y)] = 0. \tag{2}$$

Definition

The DIRAC OPERATOR of a Dirac bundle (E, c^E, ∇^E) is the essentially self-adjoint elliptic first-order differential operator D^E on E defined by

$$D^E := c^E(\nabla^E) = \sum_{i=1}^{\dim M} c^E(\varepsilon^i)\nabla^E_{e_i}, \tag{3}$$

where $\{e_1, \ldots, e_{\dim M}\}$ is any local frame for TM with dual local frame $\{\varepsilon^1, \ldots, \varepsilon^{\dim M}\}$ for T^*M.

Equivariant Dirac bundles

Let (E, c^E, ∇^E) be a G-equivariant Dirac bundle on P.

Let (E, c^E, ∇^E) be a G-equivariant Dirac bundle on P.

Question 1

How does one push forward (E, c^E, ∇^E) to a Dirac bundle $(E/G, c^{E/G}, \nabla^{E/G})$ on B?

Equivariant Dirac bundles

Let (E, c^E, ∇^E) be a G-equivariant Dirac bundle on P.

Question 1

How does one push forward (E, c^E, ∇^E) to a Dirac bundle $(E/G, c^{E/G}, \nabla^{E/G})$ on B?

Question 2

Can one isolate a HORIZONTAL DIRAC OPERATOR D^E_H, a VERTICAL DIRAC OPERATOR D^E_V, and a CURVATURE TERM Z^E, such that

1. $D^E - Z^E = D^E_V + D^E_H$,
2. $(D^E - Z^E) \circ \pi^* = D^E_H \circ \pi^* = \pi^* \circ D^{E/G}$,
3. D^E_V is a creature of the G-equivariance of (E, c^E)?

Refining the setup

Assumption

The metric $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$ restricts to an Ad-invariant inner product on $\mathfrak{g}$ over each fibre.

Refining the setup

Assumption

The metric $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$ restricts to an Ad-invariant inner product on $\mathfrak{g}$ over each fibre.

Let $V := (p \mapsto \mathrm{Vol}(G \cdot p)) \in C^\infty(P)^G$; it turns out that $\kappa = d \log V$.

Assumption

The metric $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$ restricts to an Ad-invariant inner product on $\mathfrak{g}$ over each fibre.

Let $V := (p \mapsto \mathrm{Vol}(G \cdot p)) \in C^\infty(P)^G$; it turns out that $\kappa = d \log V$.

Define $\mathbf{\Omega}^L \in \Omega^1_H(P; \mathbf{C}\mathrm{l}(T^*P))^G$ by

$$\mathbf{\Omega}^L := \frac{1}{4} \sum_{j=m+1}^{n} \varepsilon^j \cdot \Omega(\cdot, e_j)^\flat, \tag{4}$$

where $\{e_j\}_{j=m+1}^n$ is any local orthonormal frame for HP with dual local orthonormal frame $\{\varepsilon^j\}_{j=m+1}^n$ for HP^*.

Pushing forward equivariant Dirac bundles

Endow $E/G \to P/G = B$ with $(\cdot,\cdot)_{E/G}$ defined by

$$\forall \xi_1, \xi_2 \in \Gamma(E/G), \quad \pi^*(\xi_1, \xi_2)_{E/G} := V \cdot (\pi^*\xi_1, \pi^*\xi_2)_E \tag{5}$$

and $c^{E/G} : \mathbf{Cl}(H^*P) \widehat{\otimes} \mathbf{Cl}_{n-m} \to \mathrm{End}(E/G)$ defined by

$$\forall \omega \in \Omega^1(B), \quad \pi^* c^{E/G}(\omega) := c^E(\pi^*\omega), \tag{6}$$

$$\forall y \in \mathbf{Cl}_{n-m}, \quad \pi^* c^{E/G}(y) := c^E(y). \tag{7}$$

Question

Any G-equivariant $\nabla^E + A$ on E induces $\nabla^{E/G}$ on E/G by

$$\forall X \in \mathfrak{X}(B), \quad \pi^* \circ \nabla^{E/G}_X := \left(\nabla^E_{\pi^*X} + A_{\pi^*X}\right) \circ \pi^*, \tag{8}$$

so when does $(E/G, c^{E/G}, \nabla^{E/G})$ define a Dirac bundle on B?

Quotients of equivariant Dirac bundles

Proposition-Definition

The QUOTIENT *of* (E, c^E, ∇^E) *is the Dirac bundle* $(E/G, c^{E/G}, \nabla^{E/G})$ *on* B*, where* $\nabla^{E/G}$ *is the connection on* E/G *induced by*

$$ {}^0\nabla^E := \nabla^E + c^E \circ \mathbf{\Omega}^L + \frac{1}{2} d \log V. \tag{9} $$

Proposition-Definition

The QUOTIENT *of* (E, c^E, ∇^E) *is the Dirac bundle* $(E/G, c^{E/G}, \nabla^{E/G})$ *on* B*, where* $\nabla^{E/G}$ *is the connection on* E/G *induced by*

$$^{0}\nabla^E := \nabla^E + c^E \circ \mathbf{\Omega}^L + \frac{1}{2} d \log V. \tag{9}$$

Remark

Any other suitable connection on E is of the form

$$A = c^E \circ \mathbf{\Omega}^L + \frac{1}{2} d \log V + B$$

for some $B \in \Omega^1 \left(P; \mathrm{End}^{+}_{\mathbf{C}\mathrm{l}(H^*P) \widehat{\otimes} \mathbf{C}\mathrm{l}_{n-m}}(E)_{\mathrm{s.a.}} \right)^G$.

A tale of two operators

Question

What's the relationship between the Dirac operator D^E of (E, c^E, ∇^E) and the Dirac operator $D^{E/G}$ of $(E/G, c^{E/G}, \nabla^{E/G})$?

A tale of two operators

Question

What's the relationship between the Dirac operator D^E of (E, c^E, ∇^E) and the Dirac operator $D^{E/G}$ of $(E/G, c^{E/G}, \nabla^{E/G})$?

Observation

Differentiation along $G \curvearrowright (E \to P)$ yields a canonical G-equivariant even metric partial connection $\partial^E : E \to VP^* \widehat{\otimes} E$.

Question

What's the relationship between the Dirac operator D^E of (E, c^E, ∇^E) and the Dirac operator $D^{E/G}$ of $(E/G, c^{E/G}, \nabla^{E/G})$?

Observation

Differentiation along $G \curvearrowright (E \to P)$ yields a canonical G-equivariant even metric partial connection $\partial^E : E \to VP^* \widehat{\otimes} E$.

Definition

The MOMENT of (E, c^E, ∇^E) is

$$\mu^E := \nabla^E_{\Pi^{VP}(\cdot)} - \partial^E_{\Pi^{VP}(\cdot)} \in \Gamma(VP^* \widehat{\otimes} \operatorname{End}(E)^+)^G. \tag{10}$$

Proposition (Ć.–Mesland)
There exists a unique $Z^E \in \Gamma(\mathrm{End}(E))^G$ *such that*

$$(D^E - Z^E) \circ \pi^* = \pi^* \circ D^{E/G}, \tag{11}$$

namely

$$Z^E := c^E(\mu^E) - c^E(c^E \circ \mathbf{\Omega}^L) - \frac{1}{2}c^E(d \log V) \in \Gamma(\mathrm{End}(E)^-_{\mathrm{s.a.}})^G. \tag{12}$$

Proposition (Ć.–Mesland)

There exists a unique $Z^E \in \Gamma(\mathrm{End}(E))^G$ *such that*

$$(D^E - Z^E) \circ \pi^* = \pi^* \circ D^{E/G}, \tag{11}$$

namely

$$Z^E := c^E(\mu^E) - c^E(c^E \circ \boldsymbol{\Omega}^L) - \frac{1}{2} c^E(d \log V) \in \Gamma(\mathrm{End}(E)^-_{\mathrm{s.a.}})^G. \tag{12}$$

Remark

We can identify $(D^E - Z^E)|_{\Gamma(E)^G}$ with $D^{E/G}$ on $\Gamma(E/G) \cong \Gamma(E)^G$.

[Insert cliché here]

Question

So, what exactly *is* $D^E - Z^E$?

Question

So, what exactly *is* $D^E - Z^E$?

- Let $\{e_i\}_{i=1}^m$ be a local frame for VP with dual local frame $\{\varepsilon^i\}_{i=1}^m$ for $V^*P := VP^*$.
- Let $\{e_j\}_{j=m+1}^n$ be a local frame for HP with dual local frame $\{\varepsilon^j\}_{j=m+1}^n$ for $H^*P := HP^*$.

Answer

The operator $D^E - Z^E$ decomposes as

$$D^E - Z^E = D^E_V + D^E_H, \tag{13}$$

where $D^E_V = \sum_{i=1}^m c^E(\varepsilon^i)\partial^E_{e_i}$ and $D^E_H = \sum_{j=m+1}^n c^E(\varepsilon^j)^0 \nabla^E_{e_j}$.

The vertical operator

Proposition-Definition

The VERTICAL DIRAC OPERATOR *of* (E, c^E, ∇^E) *is the odd, G-invariant, essentially self-adjoint operator*

$$D_V^E := \sum_{i=1}^{m} c^E(\varepsilon^i)\partial_{e_i}^E \tag{14}$$

on E, which satisfies

$$D_V^E \circ \pi^* = 0, \tag{15}$$

$$[D_V^E, \cdot]|_{C^\infty(P)} = c^E \circ \Pi^{V^*P} \circ d, \tag{16}$$

$$\forall y \in \mathbf{C}\mathrm{l}_n, \quad [D_V^E, c^E(y)] = 0. \tag{17}$$

Proposition-Definition (cf. Brüning–Kamber, 1980s)

The HORIZONTAL DIRAC OPERATOR *of* (E, c^E, ∇^E) *is the odd, G-invariant, essentially self-adjoint operator*

$$D_H^E := \sum_{j=m+1}^{n} c^E(\varepsilon^j)^0 \nabla^E_{\varepsilon_j} \tag{18}$$

on E, which satisfies

$$D_H^E \circ \pi^* = \pi^* \circ D^{E/G}, \tag{19}$$

$$[D_H^E, \cdot]|_{\Gamma(\mathbf{C}\mathrm{l}(V^*P))} = (c^E \widehat{\otimes} c^E) \circ (\Pi^{H^*P} \widehat{\otimes} \mathrm{id}_{\mathbf{C}\mathrm{l}(V^*P)}) \circ \nabla^{\mathbf{C}\mathrm{l}(V^*P)}, \tag{20}$$

$$\forall y \in \mathbf{C}\mathrm{l}_n, \quad [D_H^E, c^E(y)] = 0. \tag{21}$$

Explicit supercommutators

Proposition (Ć.–Mesland)

The supercommutator $[D_V^E, D_H^E]$ is given locally by

$$[D_V^E, D_H^E] = -\frac{1}{2}\sum_{i=1}^{m} c^E((d_H \log g^{VP})^T(\varepsilon^i))\partial_{e_i}^E; \tag{22}$$

hence, it vanishes if and only if the metric $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$ is induced by a single Ad*-invariant metric on $\mathfrak{g}$.*

Proposition (Ć.–Mesland)

The supercommutator $[D^E_V, D^E_H]$ is given locally by

$$[D^E_V, D^E_H] = -\frac{1}{2}\sum_{i=1}^{m} c^E((d_H \log g^{VP})^T(\varepsilon^i))\partial^E_{e_i}; \tag{22}$$

hence, it vanishes if and only if the metric $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$ is induced by a single Ad*-invariant metric on $\mathfrak{g}$.*

Corollary (Ć.–Mesland)

The supercommutator

$$[D^E - Z^E, D^E_V] = 2(D^E_V)^2 + [D^E_V, D^E_H]$$

is essentially self-adjoint and bounded from below.

$$D^E - Z^E = D^E_V + D^E_H$$

$$D^E_H \circ \pi^* = \pi^* \circ D^{E/G}$$

$$\begin{cases} D^E_V \circ \pi^* = 0, \quad [D^E_V, \cdot]|_{C^\infty(P)} = c^E \circ \Pi^{V^*P} \circ d, \\ [D^E_H, \cdot]|_{\Gamma(\mathbf{Cl}(V^*P))} \\ \quad = (c^E \widehat{\otimes} c^E) \circ (\Pi^{H^*P} \widehat{\otimes} \mathrm{id}_{\mathbf{Cl}(V^*P)}) \circ \nabla^{\mathbf{Cl}(V^*P)} \end{cases}$$

Revenge of the Lie action

A fly in the ointment

Question

If (E, c^E, ∇^E) is a G-equivariant multigraded spin$^{\mathbf{c}}$ Dirac bundle on P, then $(E/G, c^{E/G}, \nabla^{E/G})$ will *never* be a multigraded spin$^{\mathbf{c}}$ Dirac bundle on B—why?

A fly in the ointment

Question

If (E, c^E, ∇^E) is a G-equivariant multigraded spin$^{\mathbf{c}}$ Dirac bundle on P, then $(E/G, c^{E/G}, \nabla^{E/G})$ will *never* be a multigraded spin$^{\mathbf{c}}$ Dirac bundle on B—why?

Answer (cf. Forsyth–Rennie, 2015)

Passing from (E, c^E, ∇^E) to $(E/G, c^{E/G}, \nabla^{E/G})$ quotients out the G-action but not the VERTICAL CLIFFORD ACTION, i.e., the vestigial Clifford action of $\mathbf{C}\mathrm{l}(V^*P)/G$ on E/G.

The vertical Clifford bundle

Let $C_V := (\mathbf{C}\mathrm{l}(V^*P) \widehat{\otimes} \mathbf{C}\mathrm{l}_m)/G = \mathbf{C}\mathrm{l}(V^*P/G) \widehat{\otimes} \mathbf{C}\mathrm{l}_m$, which is a ALGEBRA BUNDLE—a locally trivial bundle of $\mathbf{Z}/2\mathbf{Z}$-graded finite-dimensional C^*-algebras.

The vertical Clifford bundle

Let $C_V := (\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_m)/G = \mathbf{C}\mathrm{l}(V^*P/G)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_m$, which is a ALGEBRA BUNDLE—a locally trivial bundle of $\mathbf{Z}/2\mathbf{Z}$-graded finite-dimensional C^*-algebras.

- The metric connection ∇^{VP} induces an algebra bundle connection ∇^{C_V} on C_V.
- Restricting c^E to $\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_m$ induces an algebra bundle morphism $c_V^{E/G} : (C_V, \nabla^{C_V}) \to (\mathrm{End}(E/G), \nabla^{\mathrm{End}(E/G)})$.

The vertical Clifford bundle

Let $C_V := (\mathbf{C}l(V^*P)\widehat{\otimes}\,\mathbf{C}l_m)/G = \mathbf{C}l(V^*P/G)\widehat{\otimes}\,\mathbf{C}l_m$, which is a ALGEBRA BUNDLE—a locally trivial bundle of $\mathbf{Z}/2\mathbf{Z}$-graded finite-dimensional C^*-algebras.

- The metric connection ∇^{VP} induces an algebra bundle connection ∇^{C_V} on C_V.
- Restricting c^E to $\mathbf{C}l(V^*P)\widehat{\otimes}\,\mathbf{C}l_m$ induces an algebra bundle morphism $c_V^{E/G} : (C_V, \nabla^{C_V}) \to (\mathrm{End}(E/G), \nabla^{\mathrm{End}(E/G)})$.

Questions

1. When is (C_V, ∇^{C_V}) associated to $(P \to B, \Pi^{VP})$?
2. When can we find (S_V, ∇^{S_V}) associated to $(P \to B, \Pi^{VP})$, such that $(C_V, \nabla^{C_V}) \cong (\mathrm{End}(S_V), \nabla^{\mathrm{End}(S_V)})$?

Bundle metrics

Answer to Question 1

When the metric g^{TP} is a BUNDLE METRIC, i.e., a single Ad-invariant inner product on $\mathfrak{g}$ induces $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$.

Bundle metrics

Answer to Question 1

When the metric g^{TP} is a BUNDLE METRIC, i.e., a single Ad-invariant inner product on $\mathfrak{g}$ induces $g^{TP}|_{VP}$ on $VP \cong \mathfrak{g} \times P$.

Thus, the G-equivariant isomorphism $VP \cong \mathfrak{g} \times P$ induces a G-equivariant algebra bundle isomorphism

$$\Phi_V : \left((\mathbf{C}\mathrm{l}(\mathfrak{g}^*) \widehat{\otimes} \mathbf{C}\mathrm{l}_m) \times_{\mathrm{Ad}^*} P, \nabla^{(\mathbf{C}\mathrm{l}(\mathfrak{g}^*) \widehat{\otimes} \mathbf{C}\mathrm{l}_m) \times_{\mathrm{Ad}^*} P} \right) \xrightarrow{\simeq} (C_V, \nabla^{C_V}),$$

where $\nabla^{(\mathbf{C}\mathrm{l}(\mathfrak{g}^*) \widehat{\otimes} \mathbf{C}\mathrm{l}_m) \times_{\mathrm{Ad}^*} P}$ is the connection associated to Π^{VP}.

Lifting the adjoint representation

Answer to Question 2

When Ad LIFTS TO Spin, i.e., there exists a lift $\widetilde{\mathrm{Ad}} : G \to \mathrm{Spin}(\mathfrak{g})$ of $\mathrm{Ad} : G \to SO(\mathfrak{g})$.

Lifting the adjoint representation

Answer to Question 2

When Ad LIFTS TO Spin, i.e., there exists a lift $\widetilde{\mathrm{Ad}} : G \to \mathrm{Spin}(\mathfrak{g})$ of $\mathrm{Ad} : G \to SO(\mathfrak{g})$.

Fix a lift $\widetilde{\mathrm{Ad}} : G \to \mathrm{Spin}(\mathfrak{g})$ and an irreducible $\mathbf{Z}/2\mathbf{Z}$-graded $*$-representation (S, c^S) of $\mathbf{C}\mathrm{l}(\mathfrak{g}^*) \widehat{\otimes} \mathbf{C}\mathrm{l}_m \cong \mathbf{C}\mathrm{l}(\mathfrak{g}^* \oplus \mathbf{R}^m)$.

Answer to Question 2

When Ad LIFTS TO Spin, i.e., there exists a lift $\widetilde{\mathrm{Ad}} : G \to \mathrm{Spin}(\mathfrak{g})$ of $\mathrm{Ad} : G \to SO(\mathfrak{g})$.

Fix a lift $\widetilde{\mathrm{Ad}} : G \to \mathrm{Spin}(\mathfrak{g})$ and an irreducible $\mathbf{Z}/2\mathbf{Z}$-graded $*$-representation (S, c^S) of $\mathbf{C}\mathrm{l}(\mathfrak{g}^*) \widehat{\otimes} \mathbf{C}\mathrm{l}_m \cong \mathbf{C}\mathrm{l}(\mathfrak{g}^* \oplus \mathbf{R}^m)$.

Proposition-Definition

The VERTICAL SPINOR BUNDLE *is* $(S_V, c_V^{S_V}, \nabla^{S_V})$, *where:*

- $S_V := S \times_{\widetilde{\mathrm{Ad}^*}} P$;
- ∇^{S_V} *is the connection associated to* Π^{VP};
- $c_V^{S_V} := c^S \circ \pi^* \circ \Phi_V : (C_V, \nabla^{C_V}) \xrightarrow{\sim} (\mathrm{End}(S_V), \nabla^{\mathrm{End}(S_V)})$.

Proposition-Definition

The REDUCED QUOTIENT *of* (E, c^E, ∇^E) *is the Dirac bundle* $(E/\!/G, c^{E/\!/G}, \nabla^{E/\!/G})$ *on* B*, where:*

- $E/\!/G := S_V^* \widehat{\otimes}_{C_V} E/G$ *with* $(\cdot,\cdot)_{E/\!/G}$ *defined by*

$$\forall \sigma_1, \sigma_2 \in \Gamma(S_V),\ \forall \xi_1, \xi_2 \in \Gamma(E/G), \quad \big(\langle\sigma_1| \widehat{\otimes}_{C_V} \xi_1, \langle\sigma_2| \widehat{\otimes}_{C_V} \xi_2\big)_{E/\!/G} := (\xi_1, |\sigma_1\rangle\langle\sigma_2|\xi_2)_{E/G}\,; \tag{23}$$

- $c^{E/\!/G} := \mathrm{id}_{S_V} \widehat{\otimes}_{C_V} c^{E/G} : \mathbf{C}\mathrm{l}(T^*B) \widehat{\otimes}\, \mathbf{C}\mathrm{l}_{n-m} \to \mathrm{End}(E/\!/G)$;
- $\nabla^{E/\!/G} := \nabla^{S_V^* \widehat{\otimes}_{C_V} E/G}$ *induced by* ∇^{S_V} *and* $\nabla^{E/G}$.

Proposition

The morphism $\Psi_V : S_V \widehat{\otimes} E//G \to E/G$ defined by

$$\forall \sigma_1, \sigma_2 \in \Gamma(S_V),\ \forall \xi \in \Gamma(E/G),$$
$$\Psi_V \left(\sigma_1 \widehat{\otimes} \left(\langle \sigma_2 | \widehat{\otimes}_{C_V} \xi\right)\right) := \left(c_V^{E/G} \circ (c_V^{S_V})^{-1}\right) (|\sigma_1\rangle\langle\sigma_2|)\, \xi \tag{24}$$

is an isomorphism of the Dirac bundles

$$(S_V \widehat{\otimes} E//G, \mathrm{id}_{S_V} \widehat{\otimes} c^{E//G}, \nabla^{S_V \widehat{\otimes} E//G}) \xrightarrow{\sim} (E/G, c^{E/G}, \nabla^{E/G}),$$

such that

$$D^{E/G} \circ \Psi_V = \Psi_V \circ \left(\mathrm{id}_{S_V} \widehat{\otimes}_{\nabla^{S_V}} D^{E//G}\right), \tag{25}$$

$$c_V^{E/G}(\cdot) \circ \Psi_V = \Psi_V \circ \left(c_V^{S_V}(\cdot) \widehat{\otimes}\, \mathrm{id}_{E//G}\right). \tag{26}$$

$$D^{E/G} \circ \Psi_V = \Psi_V \circ \left(\mathrm{id}_{S_V} \widehat{\otimes}_{\nabla^{S_V}} D^{E//G} \right)$$

$$(E/G, c^{E/G}, \nabla^{E/G}) \cong (S_V \widehat{\otimes} E//G, \mathrm{id}_{S_V} \widehat{\otimes} c^{E//G}, \nabla^{S_V \widehat{\otimes} E//G})$$

$$\begin{cases} c_V^{E/G}(\cdot) \circ \Psi_V = \Psi_V \circ \left(c_V^{S_V}(\cdot) \widehat{\otimes}\, \mathrm{id}_{E//G} \right), \\ c_V^{S_V} : (C_V, \nabla^{C_V}) \xrightarrow{\simeq} (\mathrm{End}(S_V), \nabla^{\mathrm{End}(S_V)}), \\ [D^{E/G}, \cdot]|_{\Gamma(C_V)^G} = (c^{E/G} \widehat{\otimes} c_V^{E/G}) \circ \nabla^{C_V} \end{cases}$$

At last, some unbounded KK-theory

Relevant literature

Noncommutative $\mathbf{T}$-bundles

- S. Brain, B. Mesland, and W. D. van Suijlekom (2016)

Factorising $\mathbf{T}^n$-equivariant spectral triples

- I. Forsyth and A. Rennie (2015)

Riemannian spin$^{\mathbf{c}}$ submersions in unbounded KK-theory

- J. Kaad and W. D. van Suijlekom (2016)

Spectral triples

The Dirac bundle (E, c^E, ∇^E) yields a spectral triple

$$(C^1(P) \widehat{\otimes} \mathbf{C}\mathrm{l}_n, L^2(P,E), D^E - Z^E) \in \Psi_0^G(C(P) \widehat{\otimes} \mathbf{C}\mathrm{l}(\mathbf{R}^n), \mathbf{C}),$$

so that $[D^E] = [D^E - Z^E] \in KK_n^G(C(P), \mathbf{C})$.

The Dirac bundle (E, c^E, ∇^E) yields a spectral triple

$$(C^1(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n, L^2(P,E), D^E - Z^E) \in \Psi_0^G(C(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}(\mathbf{R}^n), \mathbf{C}),$$

so that $[D^E] = [D^E - Z^E] \in KK_n^G(C(P), \mathbf{C})$.

The quotient $(E/G, c^{E/G}, \nabla^{E/G})$ yields a spectral triple

$$(C^1(B, C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}), L^2(B, E/G), D^{E/G}) \in \Psi_0^G(C(B, C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}, \mathbf{C}),$$

so that $[D^{E/G}] \in KK_{n-m}^G(C(B, C_V), \mathbf{C})$.

Remark

The isomorphism $\Gamma(E)^G \cong \Gamma(E/G)$ yields a unitary equivalence

$$\begin{aligned}(C^1(B, C_V)&\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}), L^2(B, E/G), D^{E/G}) \\ &\cong (C^1(P, \mathbf{C}\mathrm{l}(V^*P))^G\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n, L^2(P,E)^G, (D^E - Z^E)|_{\Gamma(E)^G}).\end{aligned}$$

The vertical geometry revisited

Let ${}_{C(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}F_{C(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}$ be the completion of

$$\mathcal{F} := {}_{C^1(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}C^{1,v}\left(\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n\right)_{C^1(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}$$

with respect to $\langle\cdot,\cdot\rangle_{C(B,C_V)}\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}$ defined by

$$\text{(27)} \quad \forall \omega_1, \omega_2 \in \mathcal{F}, \quad \langle \omega_1, \omega_2 \rangle_{C(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}} := \int_G g \cdot (\omega_1^* \omega_2)\; dg.$$

The vertical geometry revisited

Let ${}_{C(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}F_{C(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}$ be the completion of

$$\mathcal{F} := {}_{C^1(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}C^{1,v}\left(\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n\right)_{C^1(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}$$

with respect to $\langle\cdot,\cdot\rangle_{C(B,C_V)}\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}$ defined by

$$\forall \omega_1,\omega_2 \in \mathcal{F}, \quad \langle \omega_1,\omega_2\rangle_{C(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}} := \int_G g\cdot(\omega_1^*\omega_2)\,dg. \tag{27}$$

Let $T : \mathcal{F} \to F$ be the densely defined operator

$$T := \sum_{i=1}^{m} \varepsilon^i \cdot \partial_{e_i}^{\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}. \tag{28}$$

The vertical geometry revisited

Let ${}_{C(P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n}F_{C(B,C_V)\widehat{\otimes}\mathbf{C}\mathrm{l}_{n-m}}$ be the completion of

$$\mathcal{F} := {}_{C^1(P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n}C^{1,v}\left(\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n\right)_{C^1(B,C_V)\widehat{\otimes}\mathbf{C}\mathrm{l}_{n-m}}$$

with respect to $\langle\cdot,\cdot\rangle_{C(B,C_V)}\widehat{\otimes}\mathbf{C}\mathrm{l}_{n-m}$ defined by

$$\forall\omega_1,\omega_2\in\mathcal{F},\quad \langle\omega_1,\omega_2\rangle_{C(B,C_V)\widehat{\otimes}\mathbf{C}\mathrm{l}_{n-m}} := \int_G g\cdot(\omega_1^*\omega_2)\,dg. \tag{27}$$

Let $T:\mathcal{F}\to F$ be the densely defined operator

$$T := \sum_{i=1}^{m}\varepsilon^i\cdot\partial_{e_i}^{\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n}. \tag{28}$$

Then $(\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n, \nabla^{\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n})$ yields

$$(C^1(P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n, F, T)\in\Psi_0^G(C(P)\widehat{\otimes}\mathbf{C}\mathrm{l}_n, C(B,C_V)\widehat{\otimes}\mathbf{C}\mathrm{l}_{n-m}),$$

$$[T]\in KK_m^G(C(P),C(B,C_V)).$$

A non-constructive factorisation

Theorem (Ć.–Mesland)
In G-equivariant KK-theory,

$$[D^E] = [D^E - Z^E] = [T]\widehat{\otimes}_{C(B,C_V)}[D^{E/G}]. \tag{29}$$

Crude sketch of proof.

1. In terms of $U : \Gamma(\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}(\mathbf{R}^n))\widehat{\otimes}_{\Gamma(C_V)}\Gamma(E/G) \xrightarrow{\sim} \Gamma(E)$,

$$D^E_V = U\left(T\widehat{\otimes}\,\mathrm{id}\right)U^*, \quad D^E_H = U\left(\mathrm{id}\,\widehat{\otimes}_{\nabla^{\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}(\mathbf{R}^n)}_{\Pi^{H^*P}(\cdot)}} D^{E/G}\right)U^*.$$

2. The supercommutator $[D^E - Z, D^E_V]$ is bounded from below.
3. Apply Kučerovský's criterion. □

A universal connection

The connection $\nabla^{\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}$ yields

$$\nabla^F : C^1\left(P, \mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n\right) \to \mathcal{F}\widehat{\otimes}^{\mathrm{alg}}_{\Gamma(C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}\left(\Gamma(T^*B\widehat{\otimes}C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}\right),$$

$$\xi \mapsto \nabla^F\xi := \sum_{k=1}^{N}\sum_{l=1}^{n-m} \nabla^{\mathbf{C}\mathrm{l}(V^*P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n}_{\pi^*(\varphi_k e[k]_l)}\xi\widehat{\otimes}\varphi_k\varepsilon[k]^l,$$

a G-equivariant connection for $(C^1(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n, F, T)$, where:

- $\mathcal{U} := \{U_k\}_{k=1}^N$ is any locally trivialising atlas for $TB \to B$;
- $\{\varphi_k\}_{k=1}^N$ is any smooth partition of unity subordinate to $\mathcal{U}$;
- for each k, $\{e[k]_l\}_{l=1}^{n-m}$ is any frame for TU_k with dual frame $\{\varepsilon[k]^l\}_{l=1}^{n-m}$ for T^*U_k.

Bundle metrics revisited

Suppose that g^{TP} is a bundle metric.

Bundle metrics revisited

Suppose that g^{TP} is a bundle metric.

1. By Peter–Weyl, we can decompose F as an L^2-direct sum of FGP $C(B, C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}$-modules associated to $P \to B$.
2. By G-equivariance, the operator T is block-diagonal.
3. Since g^{TP} is a bundle metric, local trivialisations of $P \to B$ are automatically isometric, allowing the simultaneous construction of frames on each submodule in terms of "fibrewise matrix coefficients."
4. Put together, these frames yield a column-finite frame for F that realises ∇^F as a Graßmann connection.

A constructive factorisation

Theorem (Ć.–Mesland)

If g^{TP} is a bundle metric, we have a G-equivariant constructive factorisation of $(C^1(P)\widehat{\otimes}\,\mathbb{C}\mathrm{l}_n, L^2(P,E), D^E - Z^E)$ as

(30)
$$(L^2(P,E), D^E - Z^E) \cong (F, T; \nabla^F)\widehat{\otimes}_{C^1(B,C_V)\widehat{\otimes}\,\mathbb{C}\mathrm{l}_{n-m}}(L^2(B,E/G), D^{E/G}).$$

A constructive factorisation

Theorem (Ć.–Mesland)

If g^{TP} is a bundle metric, we have a G-equivariant constructive factorisation of $(C^1(P)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_n, L^2(P,E), D^E - Z^E)$ as

(30)
$$(L^2(P,E), D^E - Z^E) \cong (F, T; \nabla^F)\widehat{\otimes}_{C^1(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}(L^2(B,E/G), D^{E/G}).$$

In particular,

(31)
$$\left(D^E - Z^E\right)\circ U = U \circ \left(T\widehat{\otimes}\,\mathrm{id}_{L^2(B,E/G)} + \mathrm{id}_F\,\widehat{\otimes}_{\nabla^F} D^{E/G}\right),$$

(32)
$$D^E_V \circ U = U \circ \left(T\widehat{\otimes}\,\mathrm{id}_{L^2(B,E/G)}\right),$$

(33)
$$D^E_H \circ U = U \circ \left(\mathrm{id}_F\,\widehat{\otimes}_{\nabla^F} D^{E/G}\right),$$

where $U : F\widehat{\otimes}_{C(B,C_V)}L^2(B,E/G) \xrightarrow{\sim} L^2(P,E)$ is the unitary itself.

Remarks

1. The data $(C^1(P) \widehat{\otimes} \mathbf{C}\mathrm{l}_n, F, T; \nabla^F)$ is independent of (E, c^E, ∇^E).
2. If g^{TP} is a bundle metric, then $(C^1(P) \widehat{\otimes} \mathbf{C}\mathrm{l}_n, F, T; \nabla^F)$ depends only on $G \curvearrowright P \twoheadrightarrow B$, the principal connection Π^{VP}, and the underlying Ad-invariant inner product on $\mathfrak{g}$.
3. If VP is G-equivariantly spin$^{\mathbf{C}}$, then $[T]$ can be identified with $\pi! \in KK_m(P, B) \cong KK_m(C(P), C(B, C_V))$.

Philosophical speculation

Remarks

1. The data $(C^1(P)\widehat{\otimes}\mathbb{C}\mathrm{l}_n, F, T; \nabla^F)$ is independent of (E, c^E, ∇^E).
2. If g^{TP} is a bundle metric, then $(C^1(P)\widehat{\otimes}\mathbb{C}\mathrm{l}_n, F, T; \nabla^F)$ depends only on $G \curvearrowright P \twoheadrightarrow B$, the principal connection Π^{VP}, and the underlying Ad-invariant inner product on $\mathfrak{g}$.
3. If VP is G-equivariantly spin$^{\mathbb{C}}$, then $[T]$ can be identified with $\pi! \in KK_m(P, B) \cong KK_m(C(P), C(B, C_V))$.

Conclusion

If g^{TP} is a bundle metric, then the data $(C^1(P)\widehat{\otimes}\mathbb{C}\mathrm{l}_n, F, T; \nabla^F)$ canonically represents the principal G-bundle $P \twoheadrightarrow B$ in the framework of spectral triples.

But what about the vertical Clifford action?

Suppose, in addition, that Ad lifts to Spin.

But what about the vertical Clifford action?

Suppose, in addition, that Ad lifts to Spin.

Observe that $(S_V, c_V^{S_V}, \nabla^{S_V})$ yields

$$(C(B, C_V) \widehat{\otimes} \mathbf{C}\mathrm{l}_{n-m}, C(B, S_V) \widehat{\otimes} \mathbf{C}\mathrm{l}_{n-m}, 0) \in \Psi_0(C(B, C_V) \widehat{\otimes} \mathbf{C}\mathrm{l}_{n-m}, C(B) \widehat{\otimes} \mathbf{C}\mathrm{l}_{n-m}),$$

which represents a class in $KK_0(C(B, C_V), C(B))$.

But what about the vertical Clifford action?

Suppose, in addition, that Ad lifts to Spin.

Observe that $(S_V, c_V^{S_V}, \nabla^{S_V})$ yields

$$(C(B, C_V) \widehat{\otimes} \mathbf{Cl}_{n-m}, C(B, S_V) \widehat{\otimes} \mathbf{Cl}_{n-m}, 0) \\ \in \Psi_0(C(B, C_V) \widehat{\otimes} \mathbf{Cl}_{n-m}, C(B) \widehat{\otimes} \mathbf{Cl}_{n-m}),$$

which represents a class in $KK_0(C(B, C_V), C(B))$.

Observe, moreover, that $\nabla^{S_V} \widehat{\otimes} \mathrm{id}_{\mathbf{Cl}_{n-m}}$ yields a connection

$$\nabla^{C(B,S_V) \widehat{\otimes} \mathbf{Cl}_{n-m}} : C^1(B, S_V) \widehat{\otimes} \mathbf{Cl}_{n-m} \\ \to \left(C(B, S_V) \widehat{\otimes} \mathbf{Cl}_{n-m}\right) \widehat{\otimes}^{\mathrm{alg}}_{C^\infty(B) \widehat{\otimes} \mathbf{Cl}_{n-m}} \left(\Omega^1(B) \widehat{\otimes} \mathbf{Cl}_{n-m}\right)$$

for $(C(B, C_V) \widehat{\otimes} \mathbf{Cl}_{n-m}, C(B, S_V) \widehat{\otimes} \mathbf{Cl}_{n-m}, 0)$.

The secondary factorisation

Proposition (Ć.–Mesland)

If g^{TP} is a bundle metric and Ad *lifts to* Spin, *we have a constructive factorisation of* $(C^1(B, C_V)\widehat{\otimes}\,\mathbb{C}\mathrm{l}_{n-m}, L^2(B, E/G), D^{E/G})$:

$$(34)\quad (L^2(B, E/G), D^{E/G}) \cong$$

$$(C(B, S_V)\widehat{\otimes}\,\mathbb{C}\mathrm{l}_{n-m}, 0; \nabla^{C(B,S_V)\widehat{\otimes}\,\mathbb{C}\mathrm{l}_{n-m}})\widehat{\otimes}_{C(B)\widehat{\otimes}\,\mathbb{C}\mathrm{l}_{n-m}}(L^2(B, E//G), D^{E//G}).$$

The secondary factorisation

Proposition (Ć.–Mesland)

If g^{TP} is a bundle metric and Ad *lifts to* Spin, *we have a constructive factorisation of* $(C^1(B, C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}, L^2(B, E/G), D^{E/G})$:

$$(34) \quad (L^2(B, E/G), D^{E/G}) \cong$$

$$(C(B, S_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}, 0; \nabla^{C(B,S_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}})\widehat{\otimes}_{C(B)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}}(L^2(B, E/\!/G), D^{E/\!/G}).$$

Remarks

1. If $G = \mathbf{T}^N$, we finally recover the commutative special case of Forsyth–Rennie (2015).
2. If E is a multigraded spinor bundle, we finally recover the case of principal bundles of Kaad–Van Suijlekom (2016).

$$
\begin{aligned}
(L^2(P,E), D^E - Z^E) &= (L^2(P,E), D^E_V + D^E_H) \\
&\cong (F, T; \nabla^F) \widehat{\otimes}_{C^1(B,C_V)\widehat{\otimes}\,\mathbf{C}\mathrm{l}_{n-m}} (L^2(B, E/G), D^{E/G})
\end{aligned}
$$

$$
\begin{aligned}
&(C^1(P, \mathbf{C}\mathrm{l}(V^*P))^G \widehat{\otimes}\, \mathbf{C}\mathrm{l}_n, L^2(P,E)^G, (D^E - Z^E)|_{\Gamma(E)^G}) \\
&\qquad \cong (C^1(B, C_V) \widehat{\otimes}\, \mathbf{C}\mathrm{l}_{n-m}, L^2(B, E/G), D^{E/G})
\end{aligned}
$$

$$
\begin{cases}
D^E_V = U \left(T \widehat{\otimes}\, \mathrm{id}_{L^2(B,E/G)} \right) U^*, \\
\\
D^E_H = U \left(\mathrm{id}_F \,\widehat{\otimes}_{\nabla^F} D^{E/G} \right) U^*
\end{cases}
$$