OPTIMAL TRANSPORT: WARMING UP BY PUSHING FORWARD (5/9-13)

In the following we assume given a map $T : X \to Y$ from a space $X$ to another space $Y$.

1. Let $\mu$ be a “pre-measure” (this is non-standard notation), i.e. real-valued function $\mu$ on the space of all subset of $X$ which is additive, i.e. $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ for any two disjoint subsets $E_1$ and $E_2$. Define the push-forward of $\mu$ under $T$, denoted by $\nu := T_* \mu$, by
   $$ (T_* \mu)(F) := \mu(T^{-1}(F)) $$
   if $F$ is a subset of $Y$. Check that $\nu$ is also a pre-measure.

2. It may also be tempting to, given a pre-measure $\nu$ on $Y$ define the “pull-back” $T^* \nu$ by
   $$ (T^* \nu)(E) := \mu(T(E)) $$
   But the pull-back operator does not preserve additivity, i.e. even if $\nu$ is additive $T^* \nu$ may not be additive. Why?

3. Can you give a condition on $T$ ensuring that $T^* \nu$ preserves additivity? First give a condition which works for any $\nu$ and then give a weaker condition which depends on $\nu$.

4. Give a function $g$ on $Y$ recall that the pull-back $T^* g$ of $g$ under $T$ is the function $f$ on $X$ defined by
   $$ f(x) := g(T(x)) $$
   Check that (under suitable regularity assumptions) $\int_Y g d\nu(T_* \mu) = \int_{T^{-1}(F)} T^* g d\mu$.

5. Conversely, show that if $\nu$ is a measure on $Y$ such that
   $$ \int_F d\nu = \int_{T^{-1}(F)} d\mu T^* g $$
   for any $g$, then $\nu = T_* \mu$.

6. Let $T : X \to Y$ be a map transporting a probability measure $\mu$ on $X$ to the measure $\nu$ on $Y$, i.e. $\nu = T_* \mu$. Denote by $I \times T$ the map $X \to X \times Y$ defined by
   $$(I \times T)(x) = (x, T(x))$$
   Check that $\gamma_T := \mu_\times(I \times T)$ is indeed a transport plan between $\mu$ and $\nu$, i.e. a probability measure on $X \times X$ whose first and second marginals coincide with $\mu$ and $\nu$, respectively. Also check that $\gamma_T$ is supported on the graph of $T$ i.e. on the set of all $(x, y)$ such that $y = T(x)$.

7. Conversely, show that if $\gamma$ is a transport plan between two probability measures $\mu$ and $\nu$, such that the support of $\gamma$ is contained in the graph of a map $T$, then $\gamma = \gamma_T$.

8. Show that if $\gamma$ is a probability measure on $X \times Y$ then it has marginals $\mu$ and $\nu$ iff
   $$ \int_{X \times Y} (\phi(x) + \psi(y)) d\gamma = \int_X \phi d\mu + \int_Y \psi d\nu $$
for any pair of (measurable and integrable) functions $\phi$ and $\psi$ on $X$ and $Y$, respectively.

(9) Let $\mu$ be a probability measure on $X$ and $y_0$ a point in $Y$. Setting $\nu = \delta_{y_0}$, i.e. the Dirac mass at $y_0$, show that there is a unique transport plan $\gamma$ from $\mu$ to $\nu$. More precisely, show that $\gamma = \gamma_T$ where $T$ is the map $T(x) = y_0$ for any $x \in X$.

(10) Given a smooth strictly convex function $\phi$ on $\mathbb{R}^n$: (i.e. the Hessian matrix $\frac{\partial^2 \phi}{\partial x_i \partial x_j}$ is positive definite) define the map $T: \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = \nabla \phi(x)$ (the gradient of $\phi$ at $x$). Check that if $\mu = fdx$ and $\nu = gdy$ are measures on $\mathbb{R}^n$: with smooth densities $f$ and $g$, then

$$T_* \mu = \nu$$

if and only if

$$\det(\frac{\partial^2 \phi}{\partial x_i \partial x_j}) g(\nabla \phi) = f(x),$$