EXERCISES 3 (OPTIMAL TRANSPORTATION)

(1) Let ϕ be a finite convex function on $\mathbb{R}^n$ and assume that 0 is an interior point of the image of sub-gradient of ϕ. Show that $\phi(x)$ is proper in the sense that $\phi(x) \geq \epsilon|x| - C$ for some positive constants ϵ and C (hint: use the Legendre transform)

(2) Let ϕ and ψ be finite convex functions on$\mathbb{R}^n$. Set $\Phi := \max\{\phi, \psi\}$. Show that Φ is a finite convex function and that for any subset U

$$(\partial\phi)(E) \cup (\partial\phi)(E) \subset (\partial\Phi)(E)$$

Deduce that $MA_g(\Phi) \geq MA_g(\phi)$ and $MA_g(\Phi) \geq MA_g(\psi)$, i.e. $MA_g(\max\{\phi, \psi\}) \geq \min\{MA_g(\phi), MA_g(\psi)\}$. Draw a picture to convince yourself that this is geometrically obvious (at least in one dimension...)

(3) Show that the functional $\mathcal{F}(\phi)$ defined in the lecture notes is concave and deduce that $J(\phi)$ is convex

(4) Show that $\mathcal{F}(\phi)$ is continuous along monotone sequences in $\mathcal{C}_Y$ in the following sense: if ϕ_j is a sequence decreasing (or increasing) to ϕ in $\mathcal{C}_Y$ then

$$\mathcal{F}(\phi_j) \to \mathcal{F}(\phi)$$

(5) Use the previous exercise to give a new proof of the fact that $\mathcal{F}(\phi)$ is lower semi-continuous on $\mathcal{C}_Y$ (Hint: if ϕ_i is a sequence in $\mathcal{C}_Y$ converging to ϕ, then there is a natural way to produce a *decreasing* sequence in $\mathcal{C}_Y$ from ϕ_i with the same limit ϕ, namely $\psi_i := \sup_{k\geq i} \phi_i$)