EXERCISES 3 (OPTIMAL TRANSPORTATION)

(1) Let $\phi$ be a finite convex function on $\mathbb{R}^n$ and assume that 0 is an interior point of the image of sub-gradient of $\phi$. Show that $\phi(x)$ is proper in the sense that $\phi(x) \geq \epsilon |x| - C$ for some positive constants $\epsilon$ and $C$ (hint: use the Legendre transform)

(2) Let $\phi$ and $\psi$ be finite convex functions on $\mathbb{R}^n$. Set $\Phi := \max\{\phi, \psi\}$. Show that $\Phi$ is a finite convex function and that for any subset $U$

$$(\partial \Phi)(E) \cup (\partial \phi)(E) \subset (\partial \phi)(E)$$

Deduce that $MA_{\Phi}(\Phi) \geq MA_{\phi}(\phi)$ and $MA_{\Phi}(\Phi) \geq MA_{\psi}(\psi)$, i.e. $MA_{\Phi}(\max\{\phi, \psi\}) \geq \min\{MA_{\phi}(\phi), MA_{\psi}(\psi)\}$. Draw a picture to convince yourself that this is geometrically obvious (at least in one dimension...)

(3) Show that the functional $F(\phi)$ defined in the lecture notes is concave and deduce that $J(\phi)$ is convex

(4) Show that $F(\phi)$ is continuous along monotone sequences in $C_Y$ in the following sense: if $\phi_j$ is a sequence decreasing (or increasing) to $\phi$ in $C_Y$ then

$$F(\phi_j) \to F(\phi)$$

(5) Use the previous exercise to give a new proof of the fact that $F(\phi)$ is lower semi-continuous on $C_Y$ (Hint: if $\phi_i$ is a sequence in $C_Y$ converging to $\phi$, then there is a natural way to produce a decreasing sequence in $C_Y$ from $\phi_i$ with the same limit $\phi$, namely $\psi_i := \sup_{k \geq i} \phi_i$)