OPTIMAL TRANSPORT: EXERCISES 2 (LEGENDRE TRANSFORMS ETC)

(1) Fix $\lambda \in \mathbb{R}$: Show that under the Legendre transform the transformation $f(x) \mapsto \lambda f(x)$ corresponds to $g(y) \mapsto \lambda g(\lambda^{-1}(y))$. Deduce that the Legendre transform of a one-homogenuous function g is equal to the indicator function χ_A for some subset A (depending on g).

(2) Fix $a \in \mathbb{R}^n$. Show that under the Legendre transform the transformation $f(x) \mapsto f(x-a)$ corresponds to $g(y) \mapsto g(y) + \langle y, a\rangle$

(3) Show that the indicator function χ_A is (a) convex iff A is convex and (b) lower semi-continuous (lsc) iff A is closed

(4) Show that (a) the Legendre transform reverses the order relation, i.e. $f \leq g \implies f^* \geq g^*$ and (b) $(f-C)^* = f^* + C$ for any constant C.

(5) Show that the following variant of the Fenchel-Rockefeller duality theorem holds in finite dimensions

$$\inf(f-g) = -\sup(f^* - g^*)$$

if f and g are convex and lower semi-continuous functions *(hint: use the previous exercise to get one inequality and then apply the argument again using the fact that $f^{**} = f$)*. In particular, show that, combined with the Fenchel-Rockefeller duality theorem one gets the following general property

$$\inf(f + \frac{1}{\beta}g) = -\frac{1}{\beta}\sup(\beta f^* - g^*(\beta \cdot))$$

for any fixed $\beta \in \mathbb{R}$.

(6) Consider $\mathbb{R}^N$ with coordinates v_i and its dual space identified with $\mathbb{R}^N$ with coordinates w_i. Define "(minus) the Gibbs entropy" of $D(v)$ by

$$D(v) = \sum v_i \log v_i$$

if $v_i \geq 0$ and $\sum_{i=1}^N v_i = 1$ and $D(v) = \infty$ otherwise. Prove that the Legendre transform of $D(v)$ is the following function:

$$L(w) := \log\left(\sum_{i=1}^{N} e^{w_i}\right)$$

(give two proofs, one involving Jensen's inequality and the other calculus).

(7) More generall, let X be a compact topological space and V the space of all signed measures on X. Identify, as usual, the topological dual V^* of V with the space $C^0(X)$. Fix a probability measure μ_0 on X and define the function $D_{\mu_0}(\mu)$ (the "entropy of μ relative to μ_0") by

$$D_{\mu_0}(\mu) = \int_X \log(\frac{d\mu}{d\mu_0})d\mu$$

if μ is a probability measure which is absolutely continuous wrt μ_0 (with density $\frac{d\mu}{d\mu_0}$) and $D_{\mu_0}(\mu) = \infty$ otherwise. Prove that the Legendre transform of $D_{\mu_0}(\mu)$ is the following functional on $C^0(X)$:

$$L_{\mu_0}(u) := \log \int_X e^u d\mu_0$$

(give two proofs, one involving Jensen's inequality and the other calculus).

(8) (the Gibbs variational principle in thermodynamics). Continuing with the notation in the previous exercise let $H(x)$ be a continuous function on X and define the corresponding linear "energy functional" $E(\mu) := \int H(x)d\mu$ if μ is a probability measure, i.e. if $\mu \in \mathcal{P}(\mu)$ and $E(\mu) = \infty$ otherwise. Show that

$$\inf_{\mu \in \mathcal{P}(\mu)} (E(\mu) + D_{\mu_0}(\mu)) = -\log \int_X e^{-H(x)} d\mu_0$$

More precisely, give two proofs: the first using Jensen's inequality and the second one using the Fenchel-Rockefeller theorem (and the previous exercise). Also, show that the minimizer of the functional $E(\mu) + D_{\mu_0}(\mu)$ is the probability measure $e^{-H} d\mu_0 / \int e^{-H} d\mu_0$ (the, so called, Gibbs measure associated to H and μ_0)

(9) First prove the following special case of the min-max principle using a calculus argument: let $F(x, y)$ be a smooth function which is convex in x and concave in y and moreover proper in each variable (so that the sup and inf are attained). Then

$$\inf_x (\sup_y F(x, y)) = (\sup_y \inf_x F(x, y))$$

Next, show that the general case in finite dimension follows by a perturbation argument.

(10) Use the Fenchel-Rockafeller theorem to prove the min-max principle (in finite dimensions).

(11) Reduce the proof of the canonical form of the linear programming duality (involving a matrix) to the special case appear in the lecture notes