Efficient solution of Helmholtz equation with applications in medical imaging

Master's Thesis Project

Background

In the project we will develop reliable and efficient algorithms and software for solution of Helmholtz equation in two and three dimensions. The problem of numerical solution of Helmholtz equation arises in many electromagnetic applications including biomedical imaging and specifically, microwave biomedical imaging. Microwave imaging is a complement to the X-ray technique with the goal to produce images of internal structure of body tissues.

This Master's project is motivated by the major challenges in microwave imaging for the medical diagnostics, such as

- high contrast reconstruction of malign tumors
- detection of small blood clots in early stages of stroke.

The above mentioned problems are modeled by the system of Maxwell's equations. Particularly, the Helmholtz equation is used by engineers as a mathematical model in microwave imaging.

The project will have close connection to the research conducted by the group of biomedical imaging at the Department of Electrical Engineering at Chalmers, see Figure 1 for different research directions where the current Master project can have applications. It is expected that application of the obtained software will be for fast detection of tumors using microwaves, see Fig. 1.

Description of the project

In this project we will consider efficient implementation of the solution of Helmholtz equation

$$\begin{aligned} &\triangle E + \omega^2 \varepsilon' E = -i\omega\mu_o J, \\ &\lim_{|x|\to\infty} E(x,\omega) = 0 \end{aligned} \tag{1}$$

in the non-magnetic medium in two and three dimensions. Here, ε' is the spatially distributed complex dielectric function $\varepsilon'(x)$:

$$\varepsilon'(x) = \varepsilon_r(x)\frac{1}{c^2} + i\mu_0 \frac{\sigma(x)}{\omega}, \tag{2}$$

where ω is the angular frequency. In (2), the functions $\varepsilon_r(x) = \varepsilon(x)/\varepsilon_0,\ \mu = \mu_r\mu_0 := \mu_0$ (since $\mu_r = 1$) and $\sigma(x)$ are relative dielectric permittivity, permeability and electric conductivity functions, respectively, and $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in free space. Finally, in the right hand side of (1) the function $J = J(x,\omega)$ is electric current density and is a known function.

We will consider finite difference discretization of Helmholtz equation in 2D and 3D. Existing C++/PETSc [8] code for solution of Helmholtz equation (1) in 2D is available for download and testing at the link [9]. Description of the code, running of it in linux in Chalmers and method of solution (FDM discretization) can be found in the Section 7 of [2].

It should be noted that the Helmholtz equation (1) is an ill-posed problem for a discrete set of frequencies ω: after finite difference discretization of this equation we obtain system of linear equations which has no solution or infinite set of solutions since the Helmholtz operator $\triangle + \omega^2\varepsilon'$ is singular. See reference [6] where is explained why the solution of (1) is difficult to obtain, specially for high frequencies. Different powerful methods are developed to circumvent these difficulties, see, for example, [4, 5] and references therein.

Purpose of the project

The main purpose of this project is to develop efficient and reliable numerical methods for solution of Helmholtz equation in two and three dimensions. The specific goals of this project are:

- Study existing numerical techniques for solution of Helmholtz equation.
- Study and modify the C++/PETSc code of [9] for different boundary conditions and different functions $\varepsilon'(x)$ related to the real-life applications (see for example [1] for values of $\varepsilon'(x)$ used in microwave hyperthermia).
- Optional: study theoretical and numerical convergence of the following methods
  - 1 - Jacobi's method,
  - 2 - Gauss-Seidel method,
  - 3 - Successive Overrelaxation method (SOR),
  - 4 - Conjugate Gradient method,
  - 5 - Conjugate Gradient method (Algorithm 12.13 of [3]),

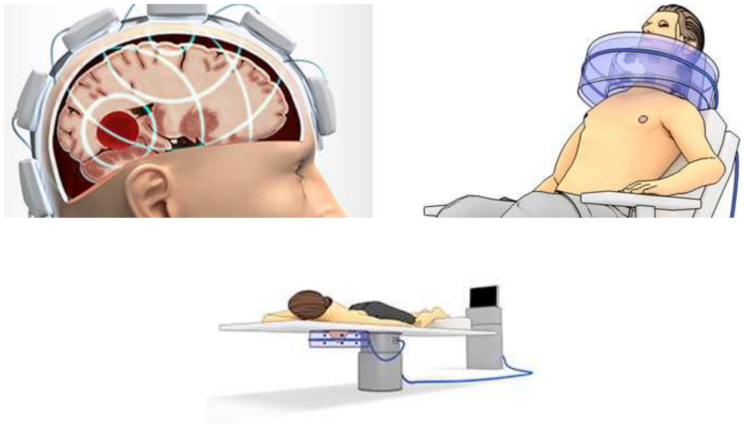

Fig. 1 Biomedical Imaging at the Department of Electrical Engineering at CTH, Chalmers. Left: setup of Stroke Finder and right: microwave hyperthermia in cancer treatment. Below: breast cancer detection using microwave tomography.

– 6 - Preconditioned Conjugate Gradient method,
– 7 - Preconditioned Conjugate Gradient method (Algorithm 12.14 of [3]).

All methods 1 - 7 listed above are already implemented in C++/PETSc, see Example 12.5 of [3] where is described the solution of the Dirichlet problem for the Poisson's equation on a unit square using iterative methods 1-7. C++/PETSc programs for solution of this problem are available for download from https://github.com/springer-math/Numerical_Linear_Algebra_Theory_and_Applications

The Master's thesis can lead further for PhD position for the project ' 'Efficient algorithms for microwave imaging based on a new non-local optimization approach", see description of the project in [7].

Contact information

Supervisor of the project: L. Beilina, larisa@chalmers.se

References

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