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Exercise 4: The double barrier – simple model for a quantum dot

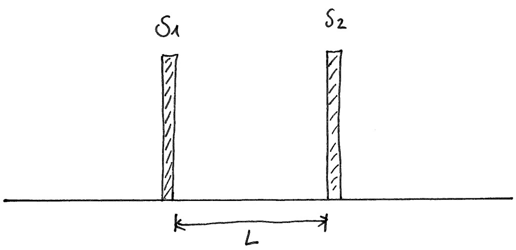

FIGURE 1 – Scatterer consisting of a double barrier (simple model for a quantum dot), where each barrier is described by a scattering matrix, S_1 and S_2..

In the lecture, we derived the conductance of a simple model of a (noninteracting) quantum dot, the double barrier, see left panel of Figure 1.

a.) Derive the scattering matrix – as obtained in the lecture – from summation of semiclassical paths between the two barriers in the conductor of length L. Take into account the different (complex) transmission and reflection amplitudes of the barriers, as well as the dynamical phase, picked up during propagation.

b.) Assuming that the transmission of the barriers is weak, derive the (approximate) expression for the transmission probability around the resonance energies E_{res} at which the transmission probability is maximal. Therefore, take the following steps :
 (1) Make an expansion for small $T_1, T_2 \ll 1$ in the denominator of the expression.
 (2) Determine the phases for which $T(E)$ is maximal and expand around these phases. Keep terms upt to second order in th small parameters T_i and $\Phi - \Phi_{\text{res}}$.
 (3) Finally, rewrite phases in terms of energies. Therefore, linearize the dispersion relation around the Fermi energy.
 You can limit your calculation to a single resonant energy, where you should find :

$$T(E) = \frac{\Gamma_1 \Gamma_2}{(E - E_{\text{res}})^2 + \frac{1}{4}(\Gamma_1 + \Gamma_2)^2} \tag{1}$$

 Which are the expressions for E_{res}, Γ_1 and Γ_2 ? How can one physically interpret these expressions ?

Exercise 5: Reminder – Properties of a Hamiltonian with vector potential

A quantum mechanical system of free, non-relativistic electrons, to which a magnetic and/or electric field is applied, can be described by the Hamiltonian

$$H = \frac{1}{2m} (\boldsymbol{p} + e\boldsymbol{A})^2 - e\phi \tag{2}$$

with the vector potential $\boldsymbol{A}$ and the scalar potential ϕ.

a.) This Hamiltonian displays the canonical momentum $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$ which is not equal to $m\boldsymbol{v}$!! Show that while the canonical momentum, $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$, still fulfills canonical commutation relations, velocities of different directions $j = x, y, z$,

$$v_j = \frac{1}{m}\left(p_j + eA_j\right) \tag{3}$$

do not commute!

b.) Consider the gauge transformation with gauge function Λ

$$\boldsymbol{A} \rightarrow \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda, \quad \phi \rightarrow \phi' = \phi - \partial_t\Lambda \tag{4}$$

Show that, if Ψ fullfills $H\Psi = i\hbar\partial_t\Psi$, the wave function

$$\tilde{\psi}(\boldsymbol{r}, t) = e^{-ie\Lambda(\boldsymbol{r},t)/\hbar}\psi(\boldsymbol{r}, t) \tag{5}$$

is a solution of the Schrödinger equation with the transformed Hamiltonian $H' = H(\boldsymbol{A} \rightarrow \boldsymbol{A}', \phi \rightarrow \phi')$.

c.) Consider now a situation, where the wave function propagates in a region, where the magnetic field is zero but the vector potential is not. What does this mean for the phase of the wave-function, when traversing a closed path?